Chapter 1 summary - Algebra and functions

1. You can simplify expressions by collecting like terms.

2. You can simplify expressions by using rules of indices (powers).

   $$a^m \times a^n = a^{m+n}$$
   $$a^m \div a^n = a^{m-n}$$
   $$a^{-m} = \frac{1}{a^m}$$
   $$a^{\frac{1}{m}} = \sqrt[m]{a}$$
   $$a^{\frac{n}{m}} = \sqrt[m]{a^n}$$
   $$(a^m)^n = a^{mn}$$
   $$a^0 = 1$$

3. You can expand an expression by multiplying each term inside the bracket by the term outside.

4. Factorising expressions is the opposite of expanding expressions.

5. A quadratic expression has the form $ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$.

6. $x^2 - y^2 = (x+y)(x-y)$

   This is called a difference of squares.

7. You can write a number exactly using surds.

8. The square root of a prime number is a surd.

9. You can manipulate surds using the rules:

   $$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$
   $$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

10. The rules to rationalise surds are:

    - Fractions in the form $\frac{1}{\sqrt{a}}$, multiply the top and bottom by $\sqrt{a}$.
    - Fractions in the form $\frac{1}{\sqrt{a+\sqrt{b}}}$, multiply the top and bottom by $a - \sqrt{b}$.
    - Fractions in the form $\frac{1}{\sqrt{a-\sqrt{b}}}$, multiply the top and bottom by $a + \sqrt{b}$.

Chapter 2 summary - Quadratic functions

1. The general form of a quadratic equation is $y = ax^2 + bx + c$ where a, b, c are constants and $a \neq 0$.

2. Quadratic equations can be solved by:

   - Factorisation.
   - Completing the square:

   $$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

   - Using the formula

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. A quadratic equation has two solutions, which may be equal.

4. To sketch a quadratic graph:

   - Decide on the shape:

   $$\begin{array}{ll} a > 0 & \cup \\ a < 0 & \cap \end{array}$$

   - Work out the *x*-axis and *y*-axis crossing points.
   - Check the general shape by considering the discriminant $b^2 - 4ac$.

Chapter 3 summary - Equations and inequalities

1. You can solve linear simultaneous equations by elimination or substitution.

2. You can use the substitution method to solve simultaneous equations, where one equation is linear and the other is quadratic. You usually start by finding an expression for x or y from the linear equation.

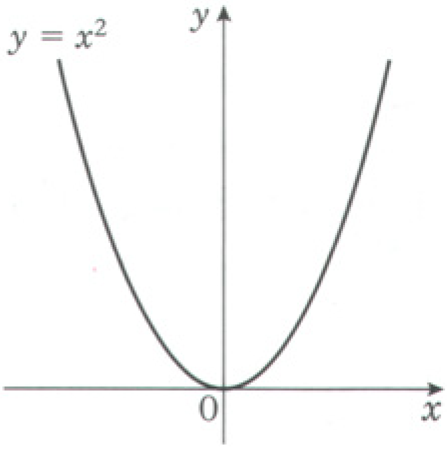
3. When you multiply or divide an inequality by a negative number, you need to change the inequality sign to its opposite.

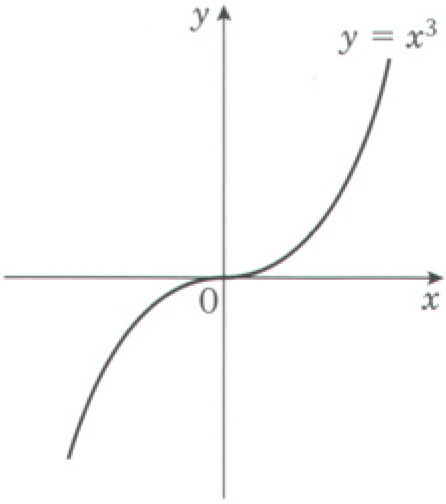
4. To solve a quadratic inequality you

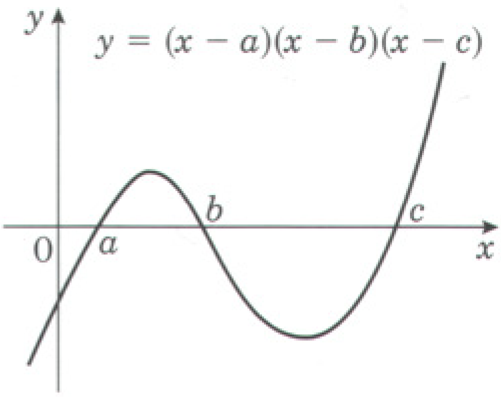
   - solve the corresponding quadratic equation, then
   - sketch the graph of the quadratic function, then
   - use your sketch to find the required set of values.

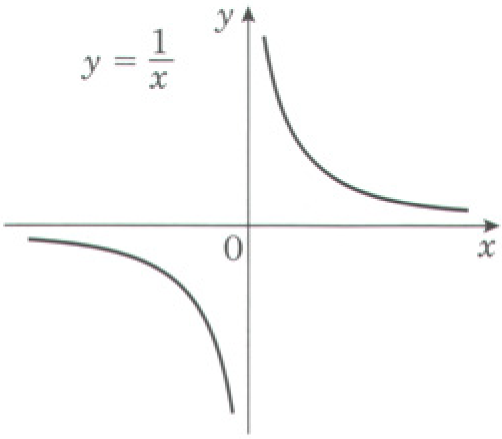
Chapter 4 summary - Sketching curves

1. You should know the shapes of the following basic curves.









2. Transformations:

   $f(x+a)$ is a translation of $-a$ in the x-direction.
   $f(x)+a$ is a translation of $+a$ in the y-direction.
   $f(ax)$ is a sketch of $\frac{1}{a}$ in the x-direction $\left(\text{multiply } x\text{-coordinates by } \frac{1}{a}\right)$.
   $af(x)$ is a sketch of a in the y-direction (multiply y-coordinates by a).

Chapter 5 summary - Coordinate geometry in the (x, y) plane

1.
   - In the general form

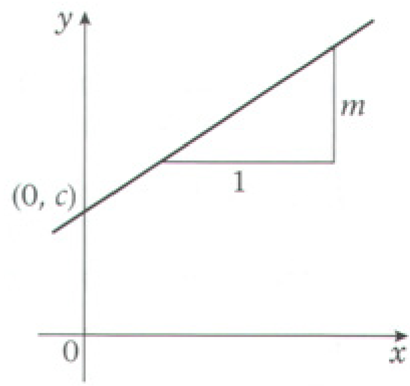
     $y = mx + c,$

     where m is the gradient and $(0, c)$ is the intercept on the y -axis.

   - In the general form

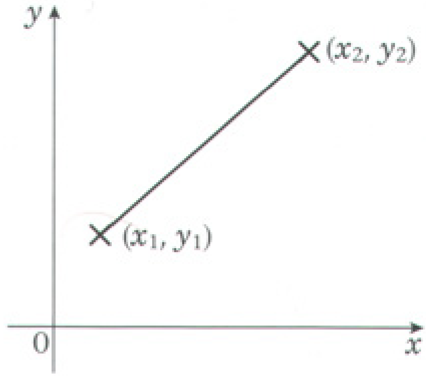
     $ax + by + c = 0,$

     where a, b and c are integers.



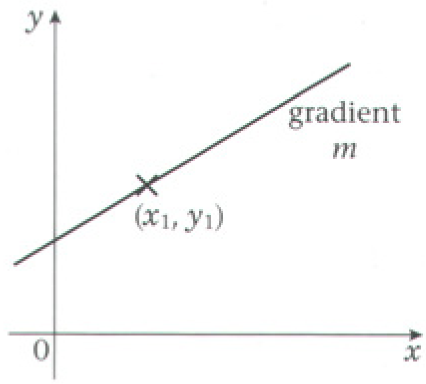
2. You can work out the gradient m of the line joining the point with coordinates (x_1, y_1) to the point with coordinates (x_2, y_2) by using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



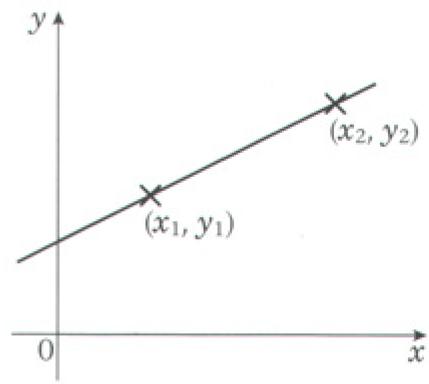
3. You can find the equation of a line with gradient m that passes through the point with coordinates (x_1, y_1) by using the formula

$$y - y_1 = m(x - x_1)$$

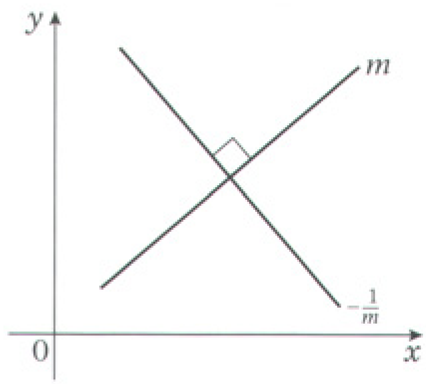


4. You can find the equation of the line that passes through the points with coordinates (x_1, y_1) and (x_2, y_2) by using the formula

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$



5. If a line has a gradient m, a line perpendicular to it has a gradient of $\frac{-1}{m}$.



6. If two lines are perpendicular, the product of their gradients is -1.

Chapter 6 summary - Sequences and series

1. A series of numbers following a set rule is called a sequence.

   3, 7, 11, 15, 19, ... is an example of a sequence.

2. Each number in a sequence is called a term.

3. The nth term of a sequence is sometimes called the general term.

4. A sequence can be expressed as a formula for the nth term. For example the formula $U_n = 4n + 1$ produces the sequence 5, 9, 13, 17, ... by replacing n with 1, 2, 3, 4, etc. in $4n + 1$.

5. A sequence can be expressed by a recurrence relationship. For example the same sequence 5, 9, 13, 17, ... can be formed from $U_{n+1} = U_n + 4, U_1 = 5$. ($U_1$ must be given.)

6. A recurrence relationship of the form

   $$U_{k+1} = U_k + n, k \geq 1 \quad n \in \mathbb{Z}$$

   is called an arithmetic sequence.

7. All arithmetic sequences can be put in the form

   $$a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + (a + 5d)$$

a	$(a+d)$	$(a+2d)$	$(a+3d)$	$(a+4d)$	$(a+5d)$
↑	↑	↑	↑	↑	↑
1st term	2nd term	3rd term	4th term	5th term	6th term

8. The nth term of an arithmetic series is $a + (n - 1)d$, where a is the first term and d is the common difference.

9. The formula for the sum of an arithmetic series is

   $$S_n = \frac{n}{2}[2a + (n - 1)d]$$

   or $S_n = \frac{n}{2}(a + L)$

   where a is the first term, d is the common difference, n is the number of terms and L is the last term in the series.

10. You can use $\sum$ to signify 'sum of'. You can use $\sum$ to write series in a more consise way

    e.g. $\sum_{r=1}^{10}(5 + 2r) = 7 + 9 + ... + 25$

Chapter 7 summary - Differentiation

1. The gradient of a curve $y = \mathrm{f}(x)$ at a specific point is equal to the gradient of the tangent to the curve at that point.
2. The gradient of the tangent at any particular point is the rate of change of y with respect to x.
3. The gradient formula for $y = \mathrm{f}(x)$ is given by the equation gradient $= \mathrm{f}'(x)$ where $\mathrm{f}'(x)$ is called the derived function.
4. If $\mathrm{f}(x) = x^n$, then $\mathrm{f}'(x) = nx^{n-1}$.

   Hint: You reduce the power by 1 and multiply the expression by the original power.

5. The gradient of a curve can also be represented by $\frac{\mathrm{d}y}{\mathrm{d}x}$.
6. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is called the derivative of y with respect to x and the process of finding $\frac{\mathrm{d}y}{\mathrm{d}x}$ when y is given is called differentiation.
7. $y = \mathrm{f}(x), \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)$
8. $y = x^n, \frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$ for all real values of n.
9. It can also be shown that if $x = ax^n$ where a is a constant, then $\frac{\mathrm{d}y}{\mathrm{d}x} = nax^{n-1}$.

   Hint: You again reduce the power by 1 and multiply the expression by the original power.

10. If $y = \mathrm{f}(x) \pm \mathrm{g}(x)$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x) \pm \mathrm{g}'(x)$.
11. A second order derivative is written as $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ or $\mathrm{f}''(x)$, using function notation.
12. You find the rate of change of a function f at a particular point by using $\mathrm{f}'(x)$ and substituting in the value of x.
13. The equation of the tangent to the curve $y = \mathrm{f}(x)$ at point $A, (a, \mathrm{f}(a))$ is $y - \mathrm{f}(a) = -\frac{1}{\mathrm{f}'(a)}(x - a)$.

Chapter 8 summary - Integration

1. If $\frac{\mathrm{d}y}{\mathrm{d}x} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + c \quad (n \neq -1)$.
2. If $\frac{\mathrm{d}y}{\mathrm{d}x} = kx^n$, then $y = \frac{kx^{n+1}}{n+1} + c \quad (n \neq -1)$.
3. $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$.

C2 summary

C2

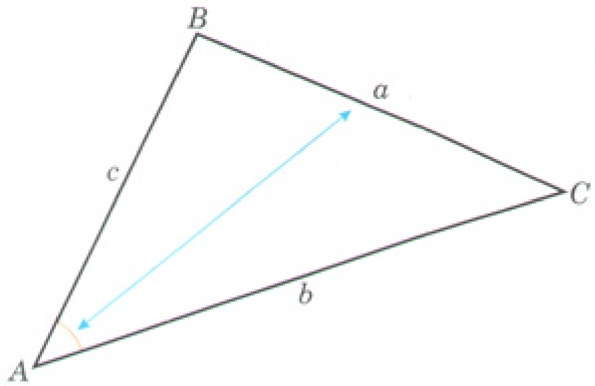
Chapter 1 summary - Algebra and functions

1. If $\mathrm{f}(x)$ is a polynomial and $\mathrm{f}(a) = 0$, then $(x - a)$ is a factor of $\mathrm{f}(x)$.
2. If $\mathrm{f}(x)$ is a polynomial and $\mathrm{f}\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $\mathrm{f}(x)$.
3. If a polynomial $\mathrm{f}(x)$ is divided by $(ax - b)$ then the remainder is $\mathrm{f}\left(\frac{b}{a}\right)$.

Chapter 2 summary - The sine and cosine rule

1. The sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



2. You can use the sine rule to find an unknown side in a triangle if you know two angles and length of one of their opposite sides.
3. You can use the sine rule to find an unknown angle in a triangle if you know the lengths of two sides and one of their opposite angles.
4. The cosine rule is

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{or} \quad b^2 = a^2 + c^2 - 2ac\cos B \quad \text{or} \quad c^2 = a^2 + b^2 - 2ab\cos C$$

5. You can use the cosine rule to find an unknown side in a triangle if you know the lengths of two sides and the angle between them.
6. You can use the cosine rule to find an unknown angle if you know the lengths of all three sides.
7. You can find an unknown angle using a rearranged form of the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \text{ or } \cos B = \frac{a^2 + c^2 - b^2}{2ac} \text{ or } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

8. You can find the area of a triangle using the formula

   area $= \frac{1}{2}ab\sin C$

   if you know the length of two sides (a and b) and the value of the angle C between them.

Chapter 3 summary - Exponentials and logarithms

1. A function $y = a^x$, or $\mathrm{f}(x) = a^x$, where a is a constant, is called an exponential function.

2. $\log_a n = x$ means that $a^x = n$, where a is called the base of the logarithm.

3. $\log_a 1 = 0$
   $\log_a a = 1$

4. $\log_{10} x$ is sometimes written as $\log x$.

5. The laws of logarithms are

   $\log_a xy = \log_a x + \log_a y$ (the multiplication law)

   $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ (the division law

   $\log_a (x)^k = k \log_a x$ (the power law)

6. From the power law,

   $$\log_a \left(\frac{1}{x}\right) = -\log_a x$$

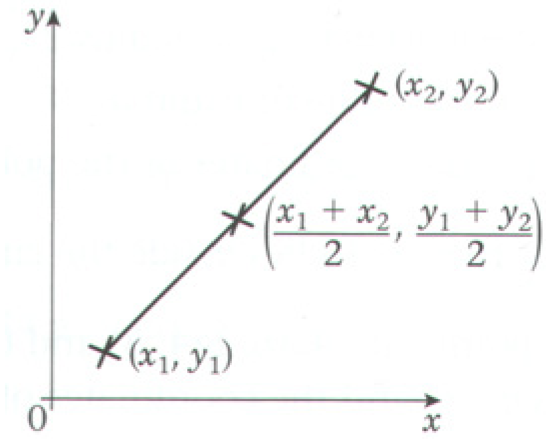
7. You can solve an equation such as $a^x = b$ by first taking logarithms (to base 10) of each side.

8. The change of base rule for logarithms can be written as $\log_a x = \dfrac{\log_b x}{\log_b a}$

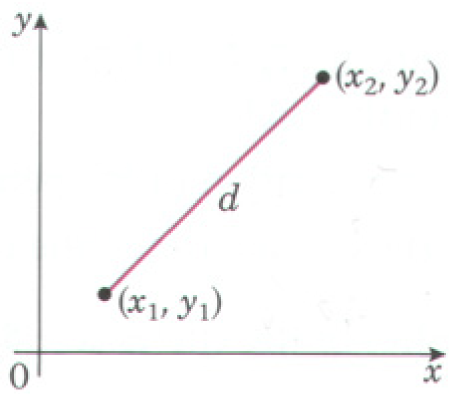
9. From the change of base rule, $\log_a b = \dfrac{1}{\log_b a}$

Chapter 4 summary - Coordinate geometry in the (x, y) plane

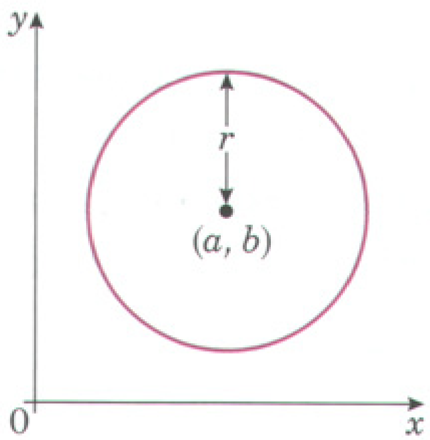
1. The mid-point of (x_1, y_1) and (x_2, y_2) is
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



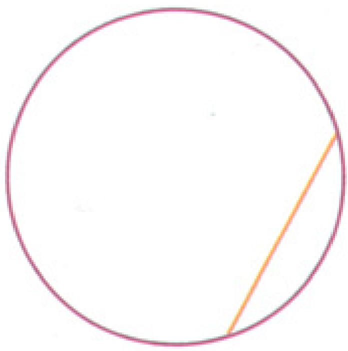
2. The distance d between (x_1, y_1) and (x_2, y_2) is
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



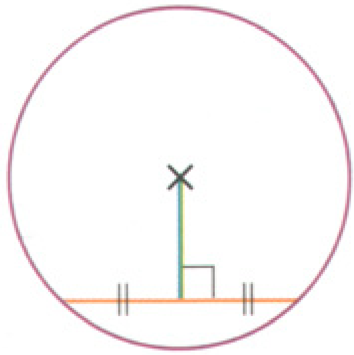
3. The equation of the circle centre (a, b) radius r is $(x - a)^2 + (y - b)^2 = r^2$.



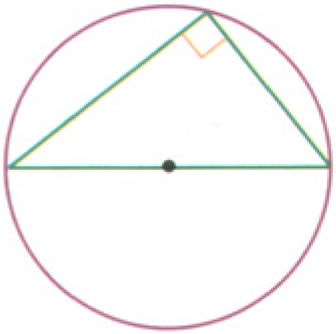
4. A chord is a line that joins two points on the circumference of a circle.



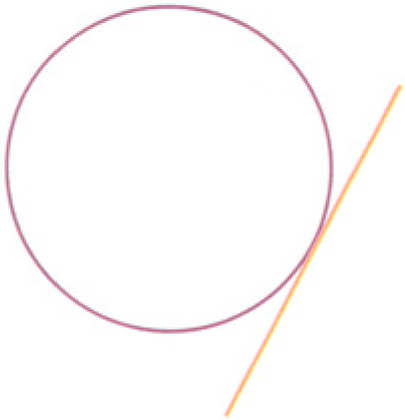
5. The perpendicular from the centre of a circle to a chord bisects the chord.



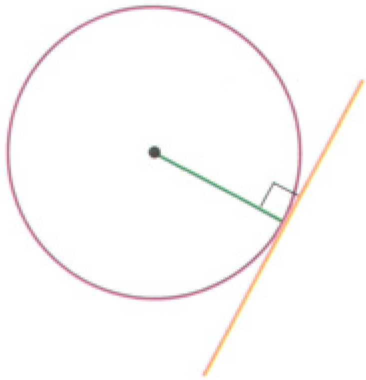
6. The angle in a semicircle is a right angle.



7. A tangent is a line that meets a circle at one point only.



8. The angle between a tangent and a radius is 90°.



Chapter 5 summary - The binomial expansion

1. You can use Pascal's Triangle to multiply out a bracket.

2. You can use combinations and factional notation to help you expand binomial expressions. For larger indices it is quicker than using Pascal's Triangle.

3. $n! = n \times (n-1) \times (n-2) \times (n-3) \times ... \times 3 \times 2 \times 1$

4. The number of ways of choosing r items from a group of n items is written ${}^nC_r$ or $\binom{n}{r}$.

   $${}^3C_2 = \frac{3!}{(3-2)!2!} = \frac{6}{1 \times 2} = 3$$

5. The binomial expansion is

   $$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + {}^nC_3 a^{n-3}b^3 + ... + {}^nC_n b^n$$

   or $\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + ... + \binom{n}{n}b^n$

6. Similarly,

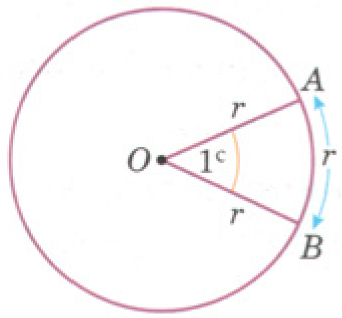
   $$(a+bx)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}bx + {}^nC_2 a^{n-2}b^2x^2 + {}^nC_3 a^{n-3}b^3x^3 + ... + {}^nC_n b^n x^n$$

   or $\binom{n}{0}a^n + \binom{n}{1}a^{n-1}bx + \binom{n}{2}a^{n-2}b^2x^2 + \binom{n}{3}a^{n-3}b^3x^3 + ... + \binom{n}{n}b^n x^n$

7. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + ...$

Chapter 6 summary - Radian measure and its applications

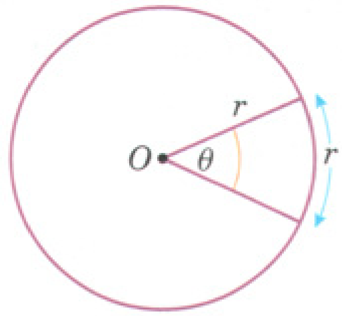
1. If the arc AB has length r, then $\angle AOB$ is 1 radian (1^c or 1 rad).



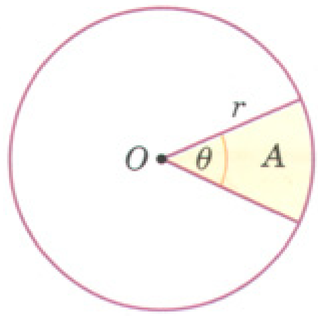
2. A radian is the angle subtended at the centre of a circle by an arc whose length is equal to that of the radius of the circle.

3. 1 radian $= \dfrac{180^\circ}{\pi}$.

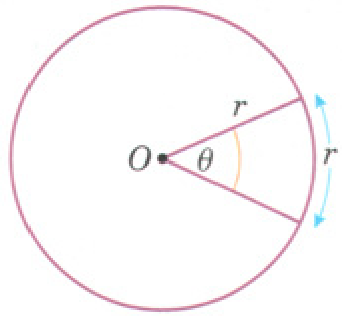
4. The length of an arc of a circle is $l = r\theta$.



5. The area of a sector is $A = \frac{1}{2}r^2\theta$.



6. The area of a segment in a circle is $A = \frac{1}{2}r^2(\theta - \sin\theta)$.



Chapter 7 summary - Geometric sequences and series

1. In a geometric series you get from one term to the next by multiplying by a constant called the common ratio.

2. The formula for the nth term $= ar^{n-1}$ where $a =$ first term and $r =$ common ratio.

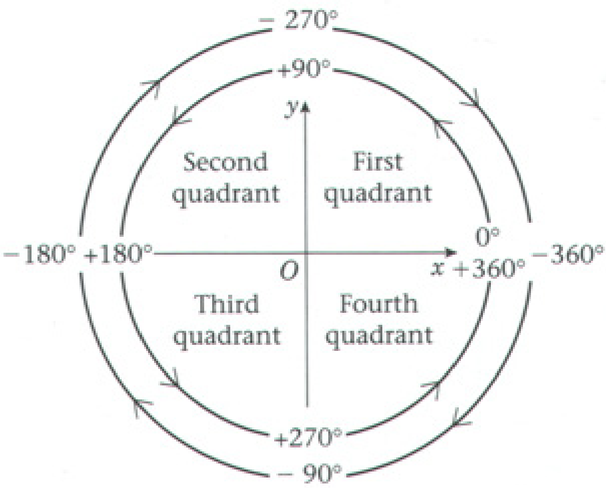
3. The formula for the sum to n terms is

   $S_n = \dfrac{a(1-r^n)}{1-r}$ or $S_n = \dfrac{a(r^n-1)}{r-1}$

4. The sum to infinity exists if $|r| < 1$ and is $S_\infty = \dfrac{a}{1-r}$

Chapter 8 summary - Graphs of trignometric functions

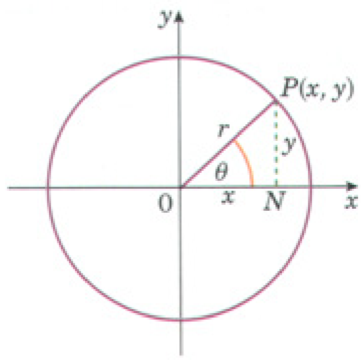
1. The x-y plane is divided into quadrants:



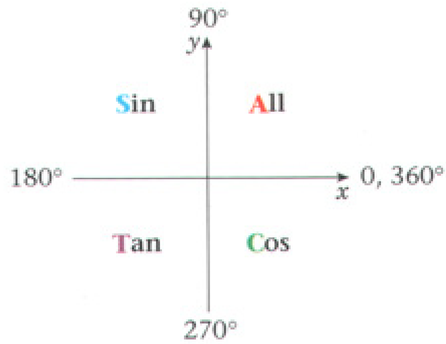
2. For all values of θ, the definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are taken to be

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

where x and y are the coordinates of P and r is the radius of the circle.

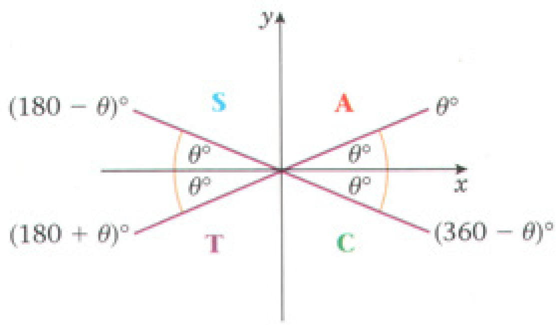


3. In the first quadrant, where θ is acute, **A**ll trigonometric functions are positive.
In the second quadrant, where θ is obtuse, only **S**ine is positive.
In the third quadrant, where θ is reflex, $180^\circ < \theta < 270^\circ$, only **T**angent is positive.
In the fourth quadrant, where θ is reflex, $270^\circ < \theta < 360^\circ$, only **C**osine is positive.



4. The trigonometric ratios of angles equally inclined to the horizontal are related:

$$\sin(180-\theta)^\circ = \sin\theta^\circ$$
$$\sin(180+\theta)^\circ = -\sin\theta^\circ$$
$$\sin(360-\theta)^\circ = -\sin\theta^\circ$$
$$\cos(180-\theta)^\circ = -\cos\theta^\circ$$
$$\cos(180+\theta)^\circ = -\cos\theta^\circ$$
$$\cos(360-\theta)^\circ = \cos\theta^\circ$$
$$\tan(180-\theta)^\circ = -\tan\theta^\circ$$
$$\tan(180+\theta)^\circ = \tan\theta^\circ$$
$$\tan(360-\theta)^\circ = -\tan\theta^\circ$$



5. The trignometric ratios of 30°, 45° and 60° have exact forms, given below:

$$\sin 30^\circ = \frac{1}{2} \qquad \cos 30^\circ = \frac{\sqrt{3}}{2} \qquad \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \tan 45^\circ = 1$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{1}{2} \qquad \tan 60^\circ = \sqrt{3}$$

6. The sine and cosine functions have a period of 360°, (or 2π radians).

   Periodic properties are

   $\sin(\theta \pm 360^\circ) = \sin\theta$ and $\cos(\theta \pm 360^\circ) = \cos\theta$

   respectively.

7. The tangent function has a period of 180°, (or π radians).

   Periodic properties is $\tan(\theta \pm 180^\circ) = \tan\theta$

8. Other useful properties are

   $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$
   $\sin(90^\circ - \theta) = \cos\theta$, $\cos(90^\circ - \theta) = \sin\theta$

Chapter 9 summary - Differentiation

1. For an increasing function $\mathrm{f}(x)$ in the interval (a, b), $\mathrm{f}'(x) > 0$ in the interval $a \leq x \leq b$.

2. For a decreasing function $\mathrm{f}(x)$ in the interval (a, b), $\mathrm{f}'(x) < 0$ in the interval $a \leq x \leq b$.

3. The points where $\mathrm{f}(x)$ stops increasing and begins to decrease are called maximum points.

4. The points where $\mathrm{f}(x)$ stops decreasing and begins to increase are called minimum points.

5. A point of inflexion is a point where the gradient is at a maximum or minimum value in the neighbourhood of the point.

6. A stationary point is a point of zero gradient. It may be a maximum, a minimum or a point of inflexion.

7. To find the coordinates of a stationary point find $\frac{\mathrm{d}y}{\mathrm{d}x}$, i.e. $\mathrm{f}'(x)$, and solve the equation $\mathrm{f}'(x) = 0$ to find the value, or values, of x and then substitute into $y = \mathrm{f}(x)$ to find the corresponding values of y .

8. The stationary value of a function is the value of y at the stationary point. You can sometimes use this to find the range of a function.

9. You may determine the nature of a stationary point by using the second derivative.

   If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, the point is a minimum point.

   If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, the point is a maximum point.

   If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, the point is either a maximum or a minumum point or a point of inflexion.

   Hint: In this case you need to use the tabular method and consider the gradient on either side of the stationary point.

   If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, but $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} \neq 0$, then the point is a point of inflexion.

10. In problems where you need to find the maximum or minimum value of a variable y, first establish a formula for y in terms of x, then differentiate and put the derived function equal to zero to find x and then y.

Chapter 10 summary - Trigonometrical identities and simple equations

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (providing $\cos\theta \neq 0$, when $\tan\theta$ is not defined)

2. $\sin^2\theta + \cos^2\theta = 1$

3. A first solution of the equation $\sin x = k$ is your calculator value, $\alpha = \sin^{-1} k$. A second solution is $(180^\circ - \alpha)$, or $(\pi - \alpha)$ if you are working in radians. Other solutions are found by adding or subtracting multiples of 360° or 2π radians.

4. A first solution of the equation $\cos x = k$ is your calculator value, $\alpha = \cos^{-1} k$. A second solution is $(360^\circ - \alpha)$, or $(2\pi - \alpha)$ if you are working in radians. Other solutions are found by adding or subtracting multiples of 360° or 2π radians.

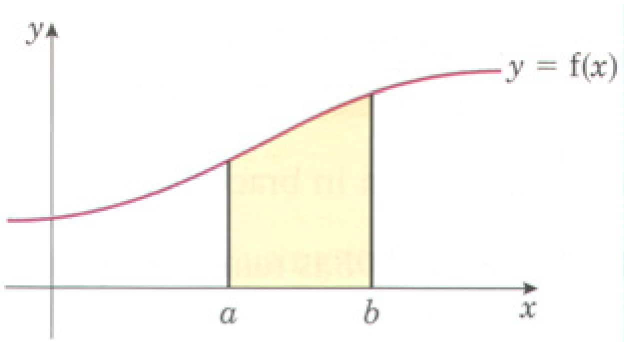
5. A first solution of the equation $\tan x = k$ is your calculator value, $\alpha = \tan^{-1} k$. A second solution is $(180^\circ + \alpha)$, or $(\pi + \alpha)$ if you are working in radians. Other solutions are found by adding or subtracting multiples of 360° or 2π radians.

Chapter 11 summary - Integration

1. The definite integral $\displaystyle\int_a^b \mathrm{f}'(x)\mathrm{d}x = \mathrm{f}(b) - \mathrm{f}(a)$.

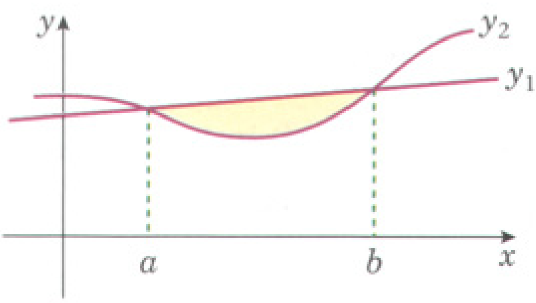
2. The area beneath the curve qith equation $y = \mathrm{f}(x)$ and between the lines $x = a$ and $x = b$ is

   Area $= \displaystyle\int_a^b \mathrm{f}(x)\mathrm{d}x$



3. The area between a line (equation y_1) and a curve (equation y_2) is given by

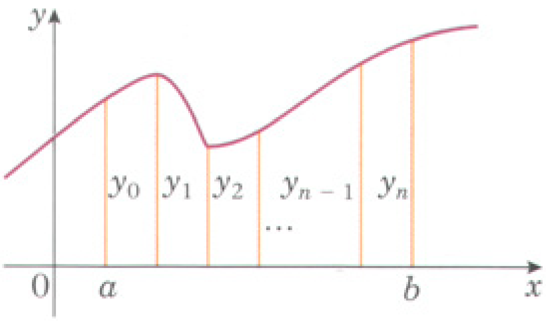
   Area $= \displaystyle\int_a^b (y_1 - y_2)\mathrm{d}x$



4. **Trapezium rule (in the formula booklet):**

$$\int_a^b y\mathrm{d}x \approx \frac{1}{2}h[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n]$$

where $h = \dfrac{b-a}{n}$ and $y_i = \mathrm{f}(a + ih)$.



Chapter 1 summary - Algebraic functions

1. Algebraic functions can be simplified by cancelling down. To do this the numerators and denominators must by fully factorised first.

2. If the numberator and denominator contain fractions then you can multiply both by the same number (the lowest common multiple) to create an equivalent fraction.

3. To multiply fractions, you simply multiply the numerators and multiply the decnominators. If possible cancel down first.

4. To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

5. To add (or subtract) fractions each fraction must have the same denominator. This is done by finding the **lowest** common multiple of the denominators.

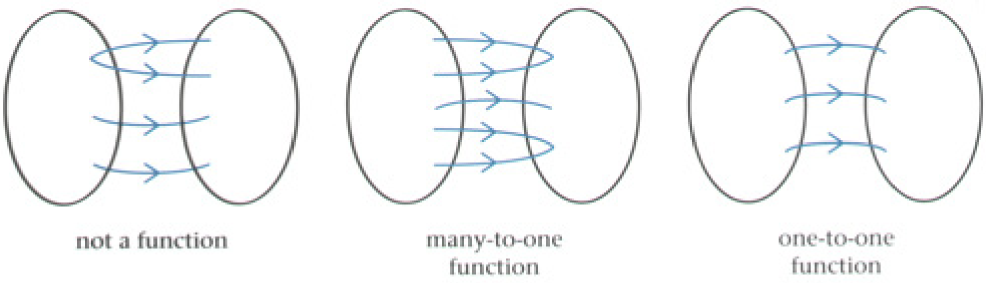
6. When the numerator hs the same or higher degree than the denominator, you can divide the terms to produce a 'mixed' number fraction. This can be done either by using long division or by using the remainded theorum:

   $\mathbf{F}(\boldsymbol{x}) \equiv \mathbf{Q}(\boldsymbol{x}) \times \text{divisor} + \text{remainder}$

   where $\mathbf{Q}(\boldsymbol{x})$ is the quotient and is how many times the divisor divides into the function.

Chapter 2 summary - Functions

1. A function is a special mapping such that every element of the domain is mapped to exactly one element in the range.



2. A one-to-one function is a special function where every element of the domain is mapped to one element in the range.

3. Many mappings can be made into functions by changing the domain. For example, the mapping 'positive square root' can be changed into the function $\mathrm{f}(x) = \sqrt{x}$ by having a domain of $x \geq 0$.

4. If we combine two or more functions we can create a composite function. The function below is written $\mathrm{fg}(x)$ as g acts on x first, then f acts on the result. For example,

   $\mathrm{g}(x) = 2x + 3, \mathrm{f}(x) = x^2$
   $\mathrm{fg}(4) = \mathrm{f}(2 \times 4 + 3) = \mathrm{f}(11) = 11^2 = 121$

   Similarly

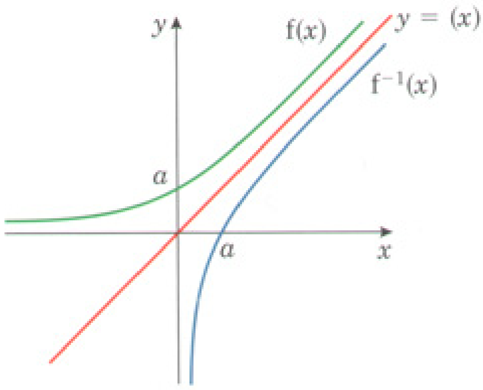
   $\mathrm{fg}(x) = (2x + 3)^2$

5. The inverse of a function $\mathrm{f}(x)$ is written $\mathrm{f}^{-1}(x)$ and performs the opposite operation(s) to the function. To calculate the inverse function you can change the subject of the formula. For example, the inverse function $\mathrm{g}(x) = 4x - 3$ is

   $$\mathrm{g}^{-1}(x) = \frac{x + 3}{4}$$

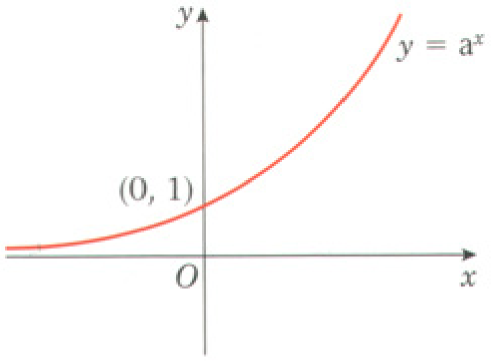
6. The range of the function is the domain of the inverse function and vice versa.

7. The graph of $\mathrm{f}^{-1}(x)$ is a reflection of $\mathrm{f}(x)$ in the line $y = x$.



Chapter 3 summary - The exponential and log functions

1. Exponential functions are ones of the form $y = a^x$.
   They all pass through the point (0,1).

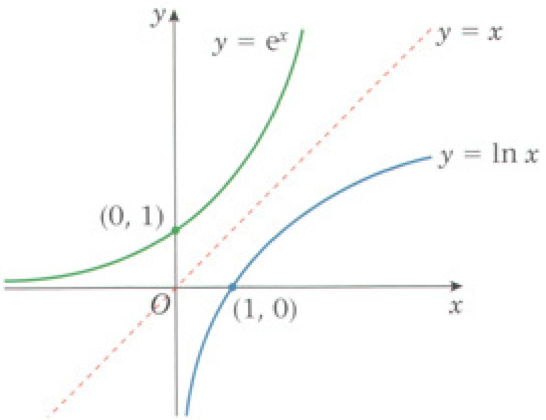
   

   The domain is all the real numbers. The range is $f(x) > 0$.

2. The exponential function $y = e^x$ (where $e \approx 2.718$) is a special function whose gradient is identical to the function.

3. The inverse function to e^x is $\ln x$.

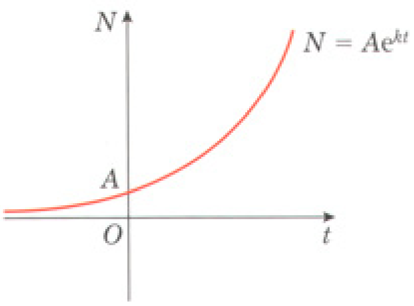
4. The natural log function is a reflection of $y = e^x$ in the line $y = x$. It passes through the point (1,0).

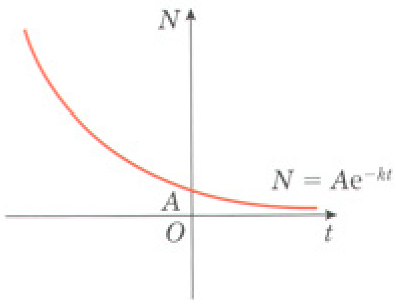
   

   The domain is the positive numbers. The range is all the real numbers.

5. To solve an equation using $\ln x$ or e^x you must change the subject of the formula and use the fact that they are inverses of each other.

6. Growth and decay models are based around the exponential equations

   where A and k are positive numbers.

Chapter 4 summary - Numerical methods

1. If you find an interval in which $f(x)$ changes sign, and $f(x)$ is continuous in that interval, then the interval must contain a root of the equation $f(x) = 0$.

2. To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into a form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$.

3. Different rearrangements of the equation $f(x) = 0$ give iteration formulae that **may** lead to different roots of the equation.

4. If you choose a value $x_0 = a$ for the starting value in an iteration formula, and $x_0 = a$ is close to a root of the equation $f(x) = 0$, then the sequence $x_0, x_1, x_2, x_3, x_4...$ does not necessarily converge to that root. In fact it might not converge to a root at all.

Chapter 5 summary - Transforming graphs of functions

1. The modulus of a number a, written as $|a|$, is its **positive** numerical value.

   - For $|a| \geq 0, |a| = a$.
   - For $|a| < 0, |a| = -a$.

2. To sketch the graph of $y = |\mathrm{f}(x)|$:

   - Sketch the graph of $y = \mathrm{f}(x)$.
   - Reflect in the x-axis any parts where $\mathrm{f}(x) < 0$ (parts below the x-axis).
   - Delete the parts below the x-axis.

3. To sketch the graph of $y = \mathrm{f}(|x|)$:

   - Sketch the graph of $y = \mathrm{f}(x)$ for $x \geq 0$.
   - Reflect this in the y-axis.

4. To solve an equation of the type $|\mathrm{f}(x)| = \mathrm{g}(x)$ or $|\mathrm{f}(x)| = |\mathrm{g}(x)|$:

   - Use a sketch to locate the roots.
   - Solve algebraically, using $-\mathrm{f}(x)$ for reflected parts of $y = \mathrm{f}(x)$ and $-\mathrm{g}(x)$ for reflected parts of $y = \mathrm{g}(x)$.

5. Basic types of transformation are

$\mathrm{f}(x+a)$	a horizontal translation of $-a$
$\mathrm{f}(x)+a$	a vertical translation of $+a$
$\mathrm{f}(ax)$	a horizontal stretch of scale factor $\frac{1}{a}$
$a\mathrm{f}(x)$	a vertical stretch of scale factor a

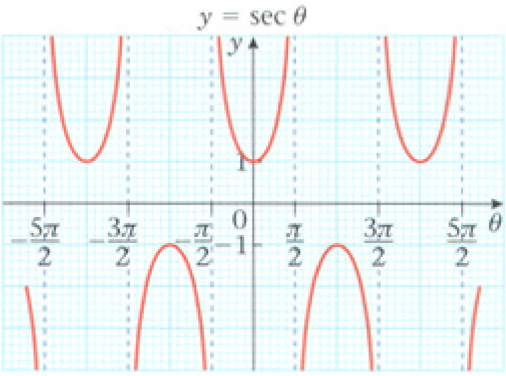
   These may be combined to give, for example, $b\mathrm{f}(x+a)$, which is a horizontal translation of $-a$ followed by a vertical stretch of scale factor b.

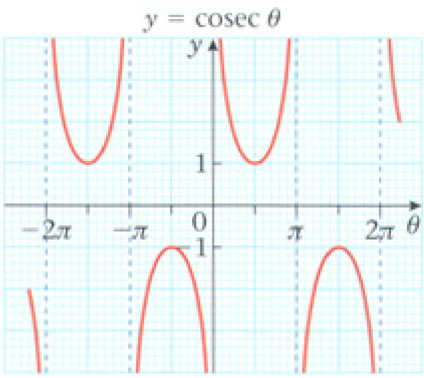
6. For combinations of transformations, the graph can be built up 'one step at a time', starting from a basic or given curve.

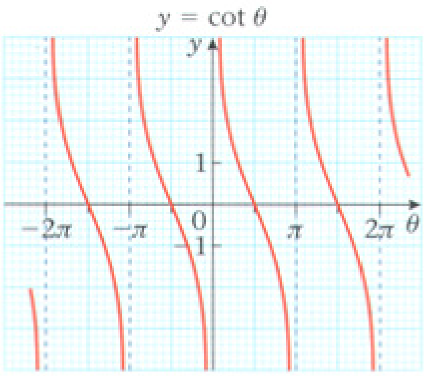
Chapter 6 summary - Trigonometry

1.
- $\sec\theta = \dfrac{1}{\cos\theta}$ ($\sec\theta$ is undefined when $\cos\theta = 0$, i.e. at $\theta = (2n+1)90^\circ, n \in \mathbb{Z}$)
- $\operatorname{co}\sec\theta = \dfrac{1}{\sin\theta}$ ($\operatorname{co}\sec\theta$ is undefined when $\sin\theta = 0$, i.e. at $\theta = 180n^\circ, n \in \mathbb{Z}$)
- $\cot\theta = \dfrac{1}{\tan\theta}$ ($\cot\theta$ is undefined when $\tan\theta = 0$, i.e. at $\theta = 180n^\circ, n \in \mathbb{Z}$)
- $\cot\theta$ can also be written as $\dfrac{\cos\theta}{\sin\theta}$.

2. The graphs of $\sec\theta$, $\operatorname{co}\sec\theta$ and $\cot\theta$ are



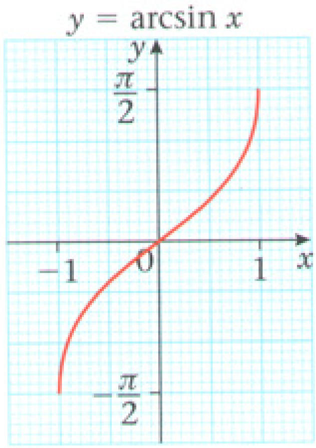




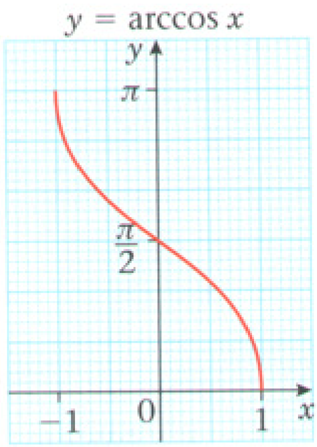
3. Two further Pythagorean identities identities, derived from $\sin^2\theta + \cos^2\theta \equiv 1$, are

$1 + \tan^2 \equiv \sec^2\theta$ and $1 + \cot^2\theta \equiv \operatorname{co}\sec\theta$

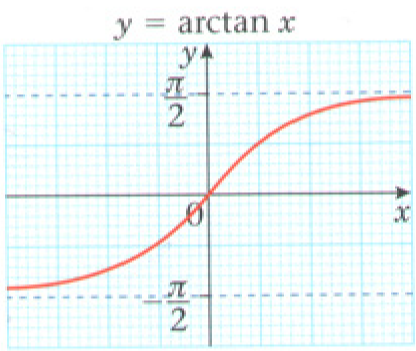
4. The inverse function of $\sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, is called $\arcsin x$; it has domain $-1 \leq x \leq 1$ and range $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$.



5. The inverse function of $\cos x$, $0 \leq x \leq \pi$, is called $\arccos x$; it has domain $-1 \leq x \leq 1$ and range $0 \leq \arccos x \leq \pi$.



6. The inverse function of $\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, is called $\arctan x$; it has domain $x \in \Re$ and range $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$.



Chapter 7 summary - Further trigonometric identities and their applications

1. The addition (or compound angle) formulae are

   - $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ $\qquad \sin(A-B) \equiv \sin A\cos B - \cos A\sin B$
   - $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ $\qquad \cos(A-B) \equiv \cos A\cos B + \sin A\sin B$
   - $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ $\qquad \tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

2. The double angle formulae are

   - $\sin 2A \equiv 2\sin A\cos A$
   - $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - \sin^2 A$
   - $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

3. Expressions of the form $a\sin\theta + b\cos\theta$ can be rewritten in terms of a sine only or a cosine only, as follows:

   For positive values of a and b,
   $a\sin\theta \pm b\cos\theta \equiv R\sin(\theta \pm \alpha)$, with $R > 0$ and $0 < \alpha < 90^\circ$,
   $a\cos\theta \pm b\sin\theta \equiv R\cos(\theta \pm \alpha)$, with $R > 0$ and $0 < \alpha < 90^\circ$

   where $R\cos\alpha = a$, $R\sin\alpha = b$ and $R = \sqrt{a^2 + b^2}$.

   Remember you can always use 'the R formula' to solve equations of the form $a\cos\theta + b\sin\theta = c$, where a, b and c are constants, but if $c = 0$, the equation reduces to the form $\tan\theta = k$.

4. Products of sines and/or cosines can be expressed as the sum or difference of sines or cosines, using the formulae:

   $$2\sin A\cos B \equiv \sin(A+B) + \sin(A-B) \qquad 2\cos A\cos B \equiv \cos(A+B) + \cos(A-B)$$

   $$2\cos A\sin B \equiv \sin(A+B) - \sin(A-B) \qquad 2\sin A\sin B \equiv -[\cos(A+B) - \cos(A-B)]$$

5. Sums or differences of sines or cosines can be expressed as a product of sines and/or cosines, using 'the factor formulae':

   $$\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) \qquad \cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P\ \ldots}{}\right.$$

   $$\sin P - \sin Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right) \qquad \cos P - \cos Q \equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P\ \ldots}{}\right.$$

Chapter 8 summary - Differentiation

You should learn all of these results.

1. You can use the chain rule to differentiate a function of a function:
   - if $y = [\mathrm{f}(x)]^n$ then $\dfrac{dy}{dx} = n[\mathrm{f}(x)]^{n-1}\mathrm{f}'(x)$
   - if $y = \mathrm{f}[\mathrm{g}(x)]$ then $\dfrac{dy}{dx} = \mathrm{f}'[\mathrm{g}(x)]\mathrm{g}'(x)$
2. Another form of the **chain rule** states that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \times \dfrac{\mathrm{d}u}{\mathrm{d}x}$ where y is a function of u, and u is a function of x.
3. A particular case of the chain rule is the result $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}$
4. You can use the product rule when two functions $u(x)$ and $v(x)$ are multiplied together.
   - If $y = uv$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}x}$
5. You can use the quotient rule when one function $u(x)$ is divided by another function $v(x)$, to form a rational function.
   - If $y = \dfrac{u}{v}$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$
6. If $y = \mathrm{e}^x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$ also and if $y = \mathrm{e}^{\mathrm{f}(x)}$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$
7. If $y = \ln x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x}$ and if $y = \ln[\mathrm{f}(x)]$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}$
8. If $y = \sin x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos x$.
9. If $y = \cos x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$
10. If $y = \tan x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec^2 x$.
11. If $y = \operatorname{co}\sec x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{co}\sec x \cot x$
12. If $y = \sec x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec x \tan x$.
13. If $y = \cot x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{co}\sec^2 x$

Chapter 1 summary - Partial Fractions

1. An algebraic fraction can be written as a sum of two or more simpler fractions. This technique is called splitting into partial fractions.

2. An expression with two linear terms in the denominator such as $\frac{11}{(x-3)(x+2)}$ can be split by converting into the form $\frac{A}{(x-3)}+\frac{B}{(x+2)}$.

3. An expression with three or more linear terms such as $\frac{4}{(x+1)(x-3)(x+4)}$ can be split by converting into the form $\frac{A}{(x+1)}+\frac{B}{(x-3)}+\frac{C}{(x+4)}$ and so on if there are more terms.

4. An expression with repeated terms in the denominator such as $\frac{6x^2-29x-29}{(x+1)(x-3)^2}$ can be split by converting into the form $\frac{A}{(x+1)}+\frac{B}{(x-3)}+\frac{C}{(x-3)^2}$.

5. An improper fraction is one where the index of the numerator is equal to or higher than the index of the denominator. An improper fraction must be divided first to obtain a number and a proper fraction before you can express it in partial fractions.

   - For example, $\frac{x^2+3x+4}{x^2+3x+2}=1+\frac{2}{x^2+3x+2}=1+\frac{A}{(x+1)}+\frac{B}{(x+2)}$.

Chapter 2 summary - Coordinate geometry in the (x, y) plane

1. To find the cartesian quation of a curve given parametrically you eliminate the parameter t between the parametric equations.

2. To find the area under a curve given parametrically you use $\int y\frac{dx}{dt}dt$

Chapter 3 summary - The Binomial Expansion

1. The binomial expansion

   $$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

   can be used to give an exact expression if n is a positive integer, or an approximate expression for any other rational number.

   - $$(1+2x)^3 = 1 + 3(2x) + 3\times 2\frac{(2x)^2}{2!} + 3\times 2\times 1\frac{(2x)^3}{3!} + 3\times 2\times 1\times 0\frac{(2x)^4}{4!}$$

     $$= 1 + 6x + 12x^2 + 8x^3$$

     (Expansion is *finite* and *exact*.)

   - $$\sqrt{(1-x)} = (1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-x) + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{(-x)^2}{2!} + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{(-x)^3}{3!} +$$

     $$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \ldots$$

     (Expansion is *infinite* and *approximate*.)

2. The expansion $(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \ldots$, where n is negative or a fraction, is only valid if $|x| < 1$.

3. You can adapt the binomial expansion to include expressions of the form $(a+bx)^n$ by simply taking out a common factor of a:

   e.g. $$\frac{1}{(3+4x)} = (3+4x)^{-1} = \left[3\left(1+\frac{4x}{3}\right)\right]$$

   $$= 3^{-1}\left(1+\frac{4x}{3}\right)^{-1}$$

4. You can use knowledge of partial fractions to expand more difficult expressions, e.g.

   $$\frac{7+x}{(3-x)(2+x)} = \frac{2}{(3-x)} + \frac{1}{(2+x)}$$

   $$= 2(3-x)^{-1} + (2+x)^{-1}$$

   $$= \frac{2}{3}\left(1-\frac{x}{3}\right)^{-1} + \frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}$$

Chapter 4 summary - Differentiation

1. When a relation is described by parametric equations:

    - You differentiate x and y with respect to the parameter t.
    - Then you use the chain rule rearranged into the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$.

2. When a relation is described by an implicit equation:

    - Differentiate each term in turn, using the chain rule and product and quotient rules as appropriate.
    - $\frac{\mathrm{d}}{\mathrm{d}x}(y^n) = ny^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}$

      By the chain rule.

    - $\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x\frac{\mathrm{d}}{\mathrm{d}x}(y) + y\frac{\mathrm{d}}{\mathrm{d}x}(x) = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$

      By the product rule.

3. In an implicit equation:

    - Note that when $\mathrm{f}(y)$ is differentiated with respect to x it becomes $\mathrm{f}'(y)\frac{\mathrm{d}y}{\mathrm{d}x}$.
    - A product term such as $\mathrm{f}(x) \cdot \mathrm{g}(y)$ is differentiated by the product rule and becomes $\mathrm{f}(x) \cdot \mathrm{g}'(y)\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{g}(y) \cdot \mathrm{f}'(x)$

4. You can differentiate the function $\mathrm{f}(x) = a^x$:

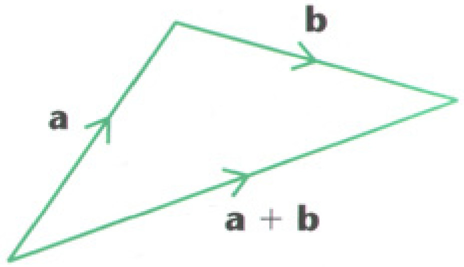
    - If $y = a^x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = a^x \ln a$

5. You can use the chain rule once, or several times, to connect the rates of change in a question involving more than two variables.

6. You can set up simple differential equations from information given in context. This may involve using connected rates of change, or ideas of proportion.

Chapter 5 summary - Vectors

1. A vector is a quantity that has both magnitude and direction.
2. Vectors that are equal have both the same magnitude and the same direction.
3. Two vectors are added using the 'triangle law'.



4. Adding the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ gives the zero vector $\mathbf{0}$.

   $\left(\overrightarrow{PQ} + \overrightarrow{PQ} = \mathbf{0}\right)$

5. The modulus of a vector is another name for its magnitude.
   - The modulus of the vector $\mathbf{a}$ is written as $|\mathbf{a}|$.
   - The modulus of the vector $\overrightarrow{PQ}$ is written as $|\overrightarrow{PQ}|$
6. The vector $-\mathbf{a}$ has the same magnitude as the vector $\mathbf{a}$ but is in the opposite direction.
7. Any vector parallel to the vector $\mathbf{a}$ may be written as $\lambda\mathbf{a}$, where λ is a non-zero scalar.
8. $\mathbf{a} - \mathbf{b}$ is defined to be $\mathbf{a} + (-\mathbf{b})$.
9. A unit vector is a vector which has magnitude (or modulus) 1 unit.
10. If $\lambda\mathbf{a} + \mu\mathbf{b} = \alpha\mathbf{a} + \beta\mathbf{b}$, and the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are not parallel, then $\lambda = \alpha$ and $\mu = \beta$.
11. The position vector of a point A is the vector $\overrightarrow{OA}$, where O is the origin. $\overrightarrow{OA}$ is usually written as vector $\mathbf{a}$.
12. $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$are the position vectors of A and B respectively.
13. The vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the x-axis, the y-axis and the z-axis and in the direction of x increasing, y increasing and z increasing, respectively.
14. The modulus (or magnitude) of $x\mathbf{i} + y\mathbf{j}$ is $\sqrt{x^2 + y^2}$.
15. The vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ may be written as a column matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
16. The distance from the origin to the point (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$.
17. The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.
18. The modulus (or magnitude) of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is $\sqrt{x^2 + y^2 + z^2}$.

19. The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is written as $\mathbf{a.b}$, and defined by

$$\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

20. If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of the points A and B, then

$$\cos AOB = \frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$$

21. The non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a.b} = 0$.

22. If $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|$.

    - In particular, $\mathbf{a.a} = |\mathbf{a}|^2$

23. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a.b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

24. A vector equation of a straight line passing through the point A with position vector $\mathbf{a}$, and parallel to the vector $\mathbf{b}$, is

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

where t is a scalar parameter.

25. A vector equation of a straight line passing through the points C and D, with position vectors $\mathbf{c}$ and $\mathbf{d}$ respectively, is

$$\mathbf{r} = \mathbf{c} + t(\mathbf{d} - \mathbf{c})$$

where t is a scalar parameter.

26. The acute angle θ between two straight lines is given by

$$\cos\theta = \left|\frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}\right|$$

where $\mathbf{a}$ and $\mathbf{b}$ are direction vectors of the lines.

Chapter 6 summary - Integration

1. You should be familiar with the following integrals.

$$\int x^n = \frac{x^{n+1}}{n+1} + C$$
$$\int \mathrm{e}^x = \mathrm{e}^x + C$$
$$\int \frac{1}{x} = \ln|x| + C$$
$$\int \cos x = \sin x + C$$
$$\int \sin x = -\cos x + C$$
$$\int \sec^2 x = \tan x + C$$
$$\int \operatorname{co}\sec \cot x = -\operatorname{co}\sec x + C$$
$$\int \operatorname{co}\sec^2 x = -\cot x + C$$
$$\int \sec x \tan x = \sec x + C$$

2. Using the chain rule in reverse you can obtain generalisations of the above formulae.

$$\int f'(ax+b)dx = \frac{1}{a}f(ax+b) + C$$
$$\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C$$
$$\int \mathrm{e}^{ax+b}dx = \frac{1}{a}\mathrm{e}^{ax+b} + C$$
$$\int \frac{1}{(ax+b)}dx = \frac{1}{a}\ln|ax+b| + C$$
$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$$
$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$$
$$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$$
$$\int \operatorname{co}\sec(ax+b)cot(ax+b)dx = -\frac{1}{a}\operatorname{co}\sec(ax+b) + C$$
$$\int \operatorname{co}\sec^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$$
$$\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + C$$

3. Sometimes trigonometric identities can be useful to help change the expression into one you know how to integrate.

e.g. To integrate $\sin^2 x$ or $\cos^2 x$ use formula for $\cos 2x$, so

$$\int \sin^2 x \mathrm{d}x = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) \mathrm{d}x$$

4. You can use partial fractions to integrate expressions of the type $\dfrac{x-5}{(x+1)(x-2)}$.

5. You should remember the following general patterns:

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \mathrm{d}x = \ln|\mathrm{f}(x)| + C$$
$$\int \mathrm{f}'(x)[\mathrm{f}(x)]^n \mathrm{d}x = \frac{1}{n+1}[\mathrm{f}(x)]^{n+1}; n \neq -1$$

6. Sometimes you can simplify an integral by changing the variable. This process is similar to using the chain rule in differentiation and is called **integration by substitution**.

7. **Integration by parts:**

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x$$

8. $\int \tan x \mathrm{d}x = \ln|\sec x| + C$

$$\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C$$
$$\int \cot x \mathrm{d}x = \ln|\sin x| + C$$
$$\int \mathrm{co}\sec x \mathrm{d}x = -\ln|\mathrm{co}\sec x + \cot x| + C$$

9. Remember: the trapezium rule is

$$\int_a^b y \mathrm{d}x \approx \frac{1}{2}h[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n] \text{ where } h = \frac{b-a}{n} \text{ and } y_i = f(a+ih).$$

10. Area of region between $y = \mathrm{f}(x)$, the x-axis and $x = a$ and $x = b$ is given by:

$$\text{Area} = \int_a^b y \mathrm{d}x$$

11. Volume of revolution formed by rotating y about the x-axis between $x = a$ and $x = b$ is given by:

$$\text{Volume} = \pi \int_a^b y^2 \mathrm{d}x$$

12. When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y)$ you can write

$$\int \frac{1}{\mathrm{g}(y)} \mathrm{d}y = \int \mathrm{f}(x) \mathrm{d}x$$

This is called separating the variables.

FP1 summary

FP1

Chapter 1 summary - Inequalities

1. Critical values of x for an inequality such as $\mathrm{f}(x) > 0$ are those values of x where the sign of $\mathrm{f}(x)$ changes for values of x on either side of the critical value.

2. Avoid multiplying an inequality by an expression which could be positive or negative.

3. When using a graphical calulator to solve an inequality, reproduce a rough sketch in your solution to illustrate your method.

4. Use a graphical approach to solve inequalities containing the modulus sign.

Chapter 2 summary - Series

1. $$\sum_{r=1}^{n} 1 = n$$

2. $$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

3. If $u_r \equiv \mathrm{f}(r+1) - \mathrm{f}(r)$, then

   $$\sum_{r=1}^{n} u_r = \mathrm{f}(n+1) - \mathrm{f}(1)$$

4. $$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

5. $$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 = \left[\sum_{r=1}^{n} r\right]^2$$

6. $$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

Chapter 3 summary - Complex numbers

1. $\sqrt{-1} = \mathrm{i}$
2. A number of the form $b\mathrm{i}$, where b is real, is called a pure imaginary number.
3. A number of the form $a + b\mathrm{i}$, where $a, b \in \Re$, is called a complex number.
4. If $z = x + \mathrm{i}y$ then the complex conjugate of z is $z^* = x - \mathrm{i}y$.
5. Any complex number can be represented by either a point or a vector on an Argand diagram.
6. If $z = x + \mathrm{i}y$ then the modulus of z is

   $$|z| = \sqrt{x^2 + y^2}$$

7. If $z = x + \mathrm{i}y$ then $\arg z$ is the principal value of the argument of z.
8. If $a + \mathrm{i}b = c + \mathrm{i}d$, where $a, b, c, d \in \Re$, then $a = c$ and $b = d$.
9. If $z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta)$

   where $-\pi \leq \theta \leq \pi$ and θ is the angle that the vector representing z on an Argand diagram makes with the positive x-axis then

   $$x = r\cos\theta \qquad y = r\sin\theta$$
   $$r = \sqrt{x^2 + y^2}$$

10. If $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$ then

    $$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2)]$$

    $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2)]$$

    $$|z_1 z_2| = |z_1||z_2|$$

    $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

    $$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

    $$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$$

11. If the polynomial equation $\mathrm{f}(z) = 0$, with real coefficients, has a root $a + b\mathrm{i}$, where $a, b \in \Re$, then the conjugate $a - b\mathrm{i}$ is also a root of the equation $\mathrm{f}(z) = 0$.

Chapter 4 summary - Numerical solution of equations

1. In order to find a root of the equation $\mathrm{f}(x) = 0$ by iteration, the equation must first be arranged in the form $x = \mathrm{g}(x)$. An iteration formula is then

   $$x_{n+1} = \mathrm{g}(x_n)$$

2. To find a root of an equation $\mathrm{f}(x) = 0$ by linear interpolation, use a straight line to join two points with x-coordinates a and b on the graph of $y = \mathrm{f}(x)$ that lie on opposite sides of the x-axis. Take the point where the line cuts the x-axis as a first approximation α_1 to the root α and work out its value by similar triangles. Work out $\mathrm{f}(\alpha_1)$ to find out whether the root lies in the interval $[a, \alpha_1]$ or $[\alpha_1, b]$ and repeat the process on the appropriate interval to find a closer approximation to α, etc.

3. Interval bisection: If a root α of the equation $\mathrm{f}(x) = 0$ lies in the interval $[a, b]$, the mid-point $\dfrac{a+b}{2}$ is a first approximation to α. Calculate $\mathrm{f}(a)$, $\mathrm{f}(b)$ and $\mathrm{f}\left(\dfrac{a+b}{2}\right)$ to find out whether the root lies in the interval $\left[a, \dfrac{a+b}{2}\right]$ or $\left[\dfrac{a+b}{2}, b\right]$ and then find the mid-point of the appropriate interval to find a close approximation to α, etc.

4. The Newton-Raphson process:
   If a is a first approximation to a root of $\mathrm{f}(x) = 0$, then

   $$a - \frac{\mathrm{f}(a)}{\mathrm{f}'(a)}$$

   is in general a better approximation.

Chapter 5 summary - First order differential equations

1. The general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y)$ is

   $$\int \frac{1}{\mathrm{g}(y)}\mathrm{d}y = \int \mathrm{f}(x)\mathrm{d}x + C$$

   provided that $\frac{1}{\mathrm{g}(y)}$ can be integrated with respect to y and $\mathrm{f}(x)$ can be integrated with respect to x. C is an arbitrary constant.

2. In the general solution of a differential equation, different values of the arbitrary constant C, arising from different initial conditions, give rise to a series of equations whose graphs when sketched are called a family of solution curves for the differential equation.

3. An exact first order differential equation is one that can be integrated directly as it stands without any processing.

4. The first order linear differential equation

   $$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$$

   where P and Q are functions of x, is made into an exact first order differential equation by multiplying the equation by the integrating factor $\mathrm{e}^{\int P\mathrm{d}x}$, provided that $\mathrm{e}^{\int P\mathrm{d}x}$ and the integral of $\mathrm{e}^{\int P\mathrm{d}x}Q(x)$ exist.

   The general solution is then

   $$y\mathrm{e}^{\int P\mathrm{d}x} = \int Q\mathrm{e}^{\int P\mathrm{d}x}\mathrm{d}x + C$$

   where C is an arbitrary constant.

Chapter 6 summary - Second order differential equations

1. For the second order differential equation

   $$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

   the auxiliary quadratic equation is

   $$am^2+bm+c=0$$

   i. If the auxiliary quadratic equation has real distinct roots α and β (condition $b^2>4ac$), then the general solution is

   $$y=Ae^{\alpha x}+Be^{\beta x}$$

   where A and B are constants.

   ii. If the auxiliary quadratic equation has real coincident roots α (condition $b^2=4ac$), then the general solution is

   $$y=(A+Bx)e^{\alpha x}$$

   where A and B are constants.

   iii. If the auxiliary quadratic equation has pure imaginary roots $\pm ni$, arising from $m^2+n^2=0$, the general solution is

   $$y=A\cos nx+B\sin nx$$

   where A and B are constants and $n\in\Re$.

   iv. If the auxiliary quadratic equation has complex cojungate roots $p\pm iq, p, q\in\Re$ (condition $b^2<4ac$), the general solution is

   $$y=e^{px}[A\cos qx+B\sin qx]$$

   where A and B are constants.

2. For the differential equation

   $$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

   where a, b and c are constants, the *complementary function* is the general solution of the differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ and a *particular integral* is any solution (i.e. function of x) that satisfies the differential equation

   $$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

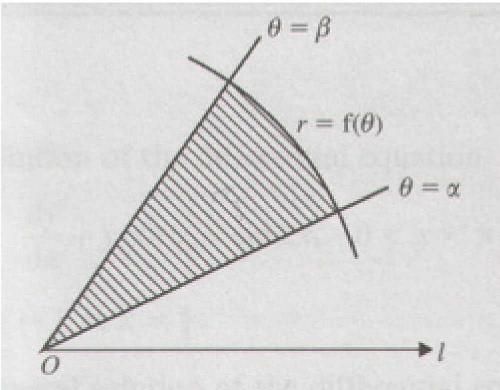
   The general solution of the differential equation is

   $y=$ complementary function + particular integral

3. A change of variable given by a substitution can transform a differential equation from one in say (x, y) which is not immediately integrable into a differential equation in say (x, v) which *is* immediately integrable or a recognised equation for which a method of solution has already been learned.

Chapter 7 summary - Polar coordinates

1. For the curve with polar equation $r = \mathrm{f}(\theta)$, area of shaded region $= \frac{1}{2}\int_{\alpha}^{\beta} r^2 \mathrm{d}\theta = \frac{1}{2}\int_{\alpha}^{\beta} [\mathrm{f}(\theta)]^2 \mathrm{d}\theta$



2. For tangents parallel to l, $\frac{\mathrm{d}}{\mathrm{d}\theta}(r\sin\theta) = 0$

   For tangents perpendicular to l, $\frac{\mathrm{d}}{\mathrm{d}\theta}(r\cos\theta) = 0$

FP3 summary

FP3

Chapter 1 summary - Maclaurin and Taylor series

1. If $y = \mathrm{f}(x)$, successive differentiation with respect to x gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x), \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{f}''(x), \ldots, \frac{\mathrm{d}^ny}{\mathrm{d}x^n} = \mathrm{f}^{(n)}(x)$$

2. Maclaurin's expansion:

$$\mathrm{f}(x) = \mathrm{f}(0) + \frac{x}{1!}\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^r}{r!}\mathrm{f}^{(r)}(0) + \ldots$$

3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots$

4. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$

5. $\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots + (-1)^{r+1} \frac{x^r}{r!} + \ldots$

6. $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots + \frac{x^r}{r!} + \ldots$

For sufficiently small x, where x^3 and higher powers of x are disregarded:

7. $\sin x \approx x \approx \tan x$

8. $\cos x \approx 1 - \frac{1}{2}x^2$

9. $(1+x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$

10. $\ln(1+x) \approx x - \frac{1}{2}x^2$

11. $\mathrm{e}^x \approx 1 + x + \frac{1}{2}x^2$

12. Taylor's series for $\mathrm{f}(x+a)$ in ascending powers of x is

$$\mathrm{f}(x+a) = \mathrm{f}(a) + x\mathrm{f}'(a) + \frac{x^2}{2!}\mathrm{f}''(a) + \ldots + \frac{x^r}{r!}\mathrm{f}^{(r)}(a) + \ldots$$

where a is a constant.

13. Taylor's series for $\mathrm{f}(x)$ in ascending powers of $(x-a)$ is

$$\mathrm{f}(x)=\mathrm{f}(a)+(x-a)\mathrm{f}'(a)+\frac{(x-a)^2}{2!}\mathrm{f}''(a)+\ldots+\frac{(x-a)^r}{r!}\mathrm{f}^{(r)}(a)+\ldots$$

where a is a constant.

14. The Taylor series solution of the differential equation $\dfrac{dy}{dx}=\mathrm{f}(x,y)$ for which $y=y_0$ at $x=x_0$ is

$$y=y_0+(x-x_0)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x_0}+\frac{(x-x_0)^2}{2!}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_{x_0}+\frac{(x-x_0)^3}{3!}\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)_{x_0}+\ldots$$

where $\left(\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}\right)_{x_0}$ is the value of $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$ at $x=x_0$.

Often $x_0=0$, and the solution is

$$y=y_0+x\left(\frac{dy}{dx}\right)_0+\frac{x^2}{2!}\left(\frac{dy}{dx}\right)_0+\frac{x^3}{3!}\left(\frac{dy}{dx}\right)_0+\ldots$$

15. An approximate value of a definite integral can sometimes be found by expanding the integrand in a series and then integrating the series term by term. This only works satisfactorily when the limits are substituted, if successive terms after the first few are rapidly getting smaller and smaller.

Chapter 2 summary - Complex numbers

1. The complex number $z = a + ib, a, b \in \Re$ can also be written as

   $z = r(\cos\theta + i\sin\theta)$ and $z = re^{i\theta}$

   where $-\pi < \theta \leq \pi$ and where $r = \sqrt{(a^2 + b^2}$ and θ is the angle which the line representing z on an Argand diagram makes with the positive x-axis.

2. $\cos iz = \cosh z$
   $\sin iz = i\sinh z$
   $\cosh iz = \cos z$
   $\sinh iz = i\sin z$

   where $z \in \mathbb{C}$.

3. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then

   $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
   $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

4. De Moivre's Theorum states that if

   $z = r(\cos\theta + i\sin\theta)$

   then

   $z^n = r^n(\cos n\theta + i\sin n\theta)$

5. The nth roots of unity are such that

   i. one root is always 1
   ii. if n is even, another root is always -1
   iii. there are n roots and if these are represented on an Argand diagram then the angle between any two consecutive roots is $\dfrac{2\pi}{n}$
   iv. the roots can be written as $1, \omega, \omega^2, \omega^3, ..., \omega^{n-2}, \omega^{n-1}$
   v. with the exception of 1 (and -1 if *n* is even) the roots occur in conjugate pairs
   vi. $1 + \omega + \omega^2 + \omega^3 + ... + \omega^{n-2} + \omega^{n-1} = 0$

6. $|z_1 + z_2| \leq |z_1| + |z_2|$
   $|z_1 + z_2| \geq \Big||z_1| - |z_2|\Big|$

Chapter 3 summary - Matrix algebra

1. A linear transformation T is such that

   $$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2)$$

   where a_1, a_2 are scalars and $\mathbf{v}_1, \mathbf{v}_2$ are vectors.

2. If T and S are linear transformations then TS is also a linear transformation and

   $$TS(\mathbf{v}) = T[S(\mathbf{v})]$$

3. If a linear transformation T is one-one then an inverse linear transformation T^{-1} exists.

4. Matrix multiplication is *not* commutative. That is, in general,

   $$\mathbf{AB} \neq \mathbf{BA}$$

5. The 2 × 2 identity matrix is $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   The 3 × 3 identity matrix is $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6. $\mathbf{AB} = \mathbf{I}$, then $\mathbf{B} = \mathbf{A}^{-1}$ and $\mathbf{A} = \mathbf{B}^{-1}$.

7. $\mathbf{I} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \dfrac{1}{ps - qr}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$

8. If $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then the transpose of $\mathbf{A}$ is

   $$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

9. Det $\mathbf{A} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

10. To find the inverse of a 3 × 3 matrix you

    i. form the matrix of minors
    ii. form the matrix of cofactors
    iii. transpose the matrix of cofactors
    iv. divide the matrix of cofactors by the determinant.

11. A matrix is called singular if its determinant is zero.

12. Only non-singular square matrices have an inverse.

13. To find a 3 × 3 matrix which represents a linear transformation T, you find $T(\mathbf{i})$, $T(\mathbf{j})$ and $T(\mathbf{k})$ and make these the first, second and third columns respectively of the matrix.

14. If $\mathbf{A}$ and $\mathbf{B}$ are two matrices then

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$
$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

15. If $\mathbf{A}$ is a matrix and $\mathbf{v}$ is a non-zero vector such that $\mathbf{Av} = \lambda\mathbf{v}$ where λ is a scalar, then $\mathbf{v}$ is called an eigenvector of $\mathbf{A}$ and λ is the corresponding eigenvalue.

16. $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is the characteristic equation of the matrix $\mathbf{A}$. The solutions of this equation given the eigenvalues of $\mathbf{A}$.

17. $\dfrac{1}{\sqrt{(a^2+b^2+c^2)}}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normalised eigenvector of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

18. The eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal if

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$$

19. The matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is orthogonal if $\begin{pmatrix} a \\ d \\ g \end{pmatrix}$, $\begin{pmatrix} b \\ e \\ h \end{pmatrix}$ and $\begin{pmatrix} c \\ f \\ i \end{pmatrix}$ are each normalised eigenvectors and if $\begin{pmatrix} a \\ d \\ g \end{pmatrix} \cdot \begin{pmatrix} b \\ e \\ h \end{pmatrix} = 0$, $\begin{pmatrix} a \\ d \\ g \end{pmatrix} \cdot \begin{pmatrix} c \\ f \\ i \end{pmatrix} = 0$ and $\begin{pmatrix} b \\ e \\ h \end{pmatrix} \cdot \begin{pmatrix} c \\ f \\ i \end{pmatrix} = 0$.

20. If $\mathbf{A}$ is orthogonal then $\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}$.

21. A square matrix whose elements are all zero except those on the leading diagonal is called a diagonal matrix.

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ are diagonal matrices.

22. A matrix $\mathbf{A}$ which has $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$ is symmetric.

23. If $\mathbf{A}$ is symmetric and $\mathbf{P}$ is an orthogonal matrix whose columns are the normalised, orthogonal eigenvectors of $\mathbf{A}$, then $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is diagonal.

Chapter 4 summary - Vectors

1. The vector product of $\mathbf{a}$ and $\mathbf{b}$ is

   $$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \hat{\mathbf{n}}$$

   where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ which is in the direction that a right-handed corkscrew would move when turned from $\mathbf{a}$ to $\mathbf{b}$.

2. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

3. i. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
   ii. $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
   iii. $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

   where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the directions of the positive x-, y- and z-axes respectively.

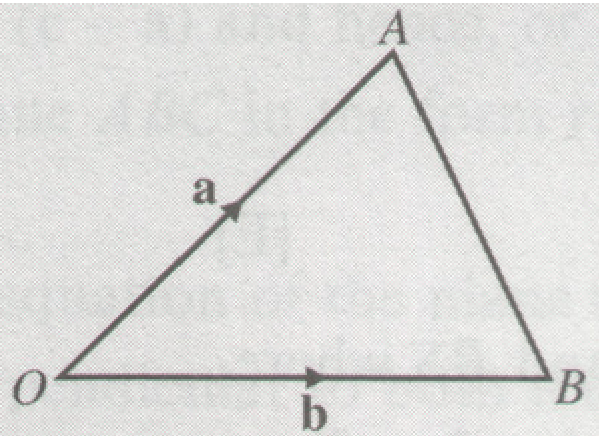
4. i. $\mathbf{i} \times \mathbf{i} = \mathbf{0}$
   ii. $\mathbf{j} \times \mathbf{j} = \mathbf{0}$
   iii. $\mathbf{k} \times \mathbf{k} = \mathbf{0}$

   where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the directions of the positive x-, y- and z-axes respectively.

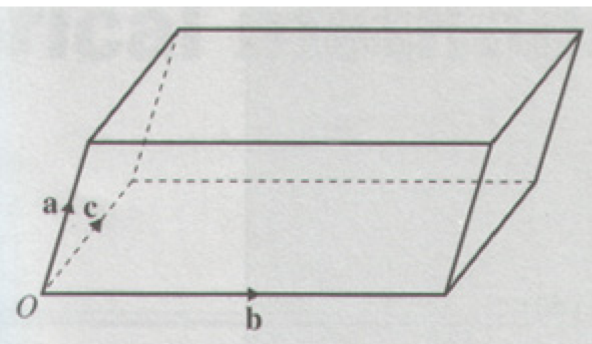
5. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$
   and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

   then $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

6. If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ then either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{a}$ are parallel.

7. 

   Area of $\triangle AOB = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

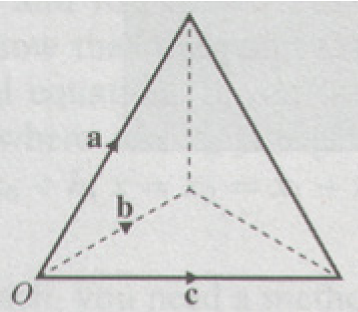
8. 

The volume of the parallelepiped is

$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$

9. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$
$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$
$\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

then $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

10. 

The volume of the tetrahedron is

$\frac{1}{6}\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$

11. The equation of the line passing through A, with position vector $\mathbf{a}$, and the point R, with position vector $\mathbf{r}$, and which is parallel to the vector $\mathbf{b}$, is

$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$

12. The equation of the plane containing the points A and R, with position vectors $\mathbf{a}$ and $\mathbf{r}$ respectively, is $\mathbf{r} \cdot \mathbf{n} = p$, where $p = \mathbf{a} \cdot \mathbf{n}$ and $\mathbf{n}$ is a vector perpendicular to the plane.

13. The vector equation of a plane passing through the point with position vector $\mathbf{a}$ and where $\mathbf{b}$ and $\mathbf{c}$ are non-parallel vectors in the plane, neither of which is zero is

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$,

where λ and μ are scalars.

14. The distance d from the origin to the plane containing the point with position vector $\mathbf{r}$ is

$d = \mathbf{r} \cdot \hat{\mathbf{n}}$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane.

15. The acute angle θ between a line, with direction vector $\mathbf{b}$, and a plane is

$\arcsin\left|\frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{b}||\mathbf{n}|}\right|$

where $\mathbf{n}$ is a vector perpendicular to the plane.

16. The acute angle θ between two planes is given by

$$\cos\theta = \left|\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|$$

where $\mathbf{n}_1$ is a vector perpendicular to one of the planes and $\mathbf{n}_2$ is a vector perpendicular to the other plane.

17. The shortest distance between the lines with equations $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$ where λ, μ are scalars is given by

$$\left|\frac{(\mathbf{a} - \mathbf{c}) \cdot \mathbf{b} \times \mathbf{d}}{|\mathbf{b} \times \mathbf{d}|}\right|$$

Chapter 5 summary - Numerical methods

1. In step-by-step methods, where the step length is h, learn the approximations and how to use them:

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$$
$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$$
$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

Chapter 6 summary - Proof

A proof by mathematical induction consists in showing that if a theorum is true for some special integral value of n, say $n = k$, then it is true for $n = k + 1$. Also you need to show that the theorum is true for some trivial value of n such as $n = 1$ (or $n = 2$, etc.). Then if it is true for $n = k + 1$, when it is true for $n = k$, and if it is true for $n = 1$, then it is true for $n = 1 + 1 = 2$, $n = 1 + 2 = 3$ and so on for all positive integral n.

Chapter 1 summary - Mathematical models in mechanics

This chapter does not have a summary.

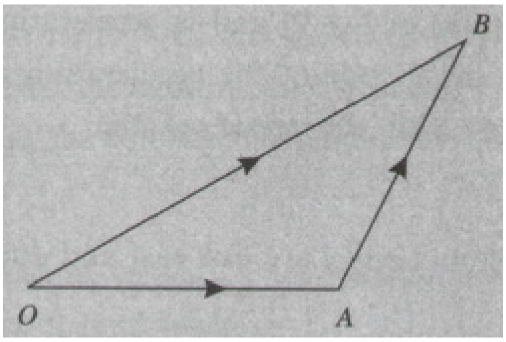
Chapter 2 summary - Vectors and their application in mechanics

1. Scalar quantities are completely specified by their magnitude.

2. Vector quantities require both their size (magnitude) and direction to be specified.

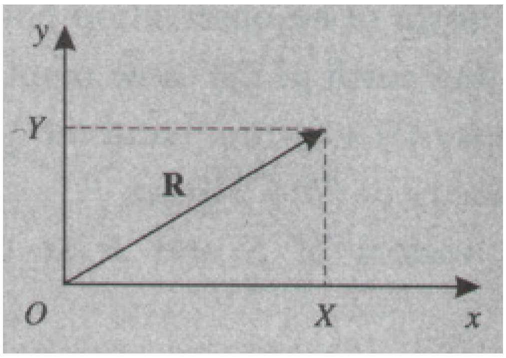
3. Vectors are added by using the triangle law of addition:

   $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

4. Using the $\mathbf{i}, \mathbf{j}$ notation:

   $\mathbf{R} = X\mathbf{i} + Y\mathbf{j}$

5. If particle A has position vector $\mathbf{r}_A$ and particle B has position vector $\mathbf{r}_B$ then the position vector of B relative to A is $\mathbf{r}_B - \mathbf{r}_A$.

6. If particle A has velocity $\mathbf{v}_A$ and particle B has velocity $\mathbf{v}_B$ then the velocity of B relative to A is $\mathbf{v}_B - \mathbf{v}_A$.

Chapter 3 summary - Kinematics of a particle

1. **Constant acceleration**
   For a particle moving with constant acceleration:

   $$v = u + at$$
   $$s = \left(\frac{u+v}{2}\right)t$$
   $$s = ut + \tfrac{1}{2}at^2$$
   $$v^2 = u^2 + 2as$$

2. **Speed-time graphs**
   The area under a speed-time graph in a given time interval is the distance travelled during that time.
   The gradient of a speed-time graph is the acceleration of the moving particle.

Chapter 4 summary - Statics of a particle

1. Force is a vector quantity.

2. Forces can be added by using the triangle law or parallelogram rule.

3. The resultant of a system of forces is most easily found by using components.

4. A system of forces is in equilibrium if their lines of action pass through a single point and if their resultant is the zero vector.

5. The magnitude of the frictional force is just sufficient to prevent relative motion.

6. For a smooth surface there is no frictional force ($F = 0$).

7. When sliding occurs the frictional force takes its limiting value μR and opposes the relative motion.

Chapter 5 summary - Dynamics of a particle moving in a straight line or plane

1. **Newton's Laws of Motion**
   1. A particle will only accelerate if it is acted on by a resultant force.
   2. The force F applied to a particle is proportional to the mass m of the particle and the acceleration produced.
      $\mathbf{F} = m\mathbf{a}$
   3. The forces between two bodies in contact are equal in magnitude but opposite in direction.

2. **Momentum and Impulse**
   The momentum of a particle of mass m moving with velocity $\mathbf{v}$ is $m\mathbf{v}$.
   If a constant force $\mathbf{F}$ acts on a particle for time t, then the impulse of the force is equal to $\mathbf{F}t$.

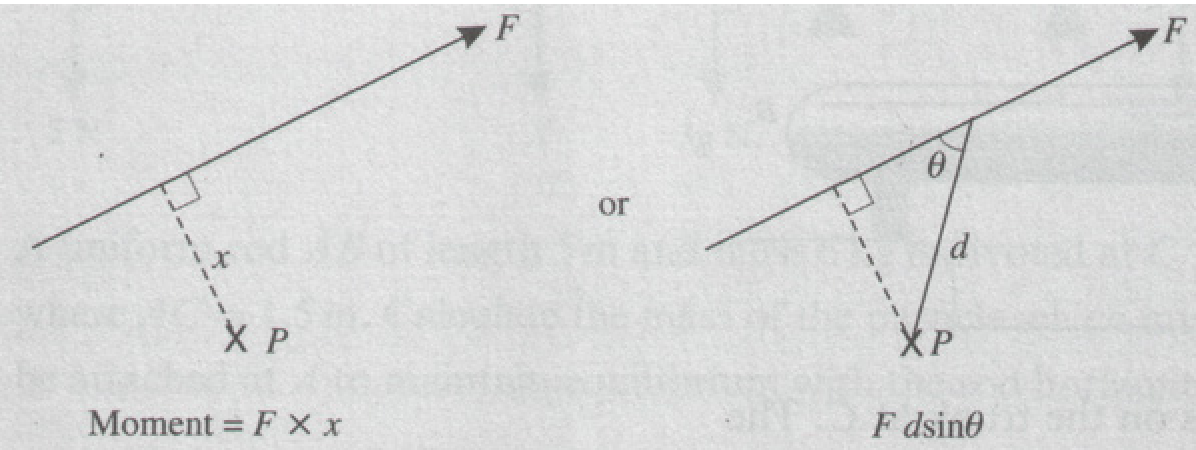
   Impulse = change in momentum

   For a force in Newtons and time in seconds, impulse and momentum are measured in Newton-seconds.

3. **Conservation of momentum**
   Momentum is conserved for two particles experiencing a collision or jerk where there are no external forces involved.
   $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

Chapter 6 summary - Moments

1. The moment of a force of magnitude F about a point P is given by:

   For a force in Newtons and a distance in metres the moment is measured in Newton-metres (Nm). Moments can be clockwise or anticlockwise in sense.

2. For a lamina to be in equilibrium under parallel forces:
   a. the resultant force in any direction must be zero
   b. the component of the resultant force in any direction must be zero
   c. the algebraic sum of the moments about any point must be zero.

3. A rod which is in equilibrium resting on two supports at the same horizontal level is said to be on the point of tilting about one support when the reation at the other support has magnitude zero.

Chapter 1 summary - Kinematics of a particle moving in a straight line or plane

1. **Projectiles**
   The horizontal speed of a projectile is unchanged throughout the motion. The vertical motion is subject to an acceleration of magnitude $g = 9.8\text{ms}^{-2}$ vertically downwards. The four uniform acceleration equations apply (see M1 summary).

2. **Variable acceleration**
   For variable acceleration the relationships between the displacement x, velocity v and acceleration a of a particle are shown by:

   Differentiate with respect to t $\Downarrow$ $\quad x \;|\; v \;|\; a \quad$ $\Uparrow$ Integrate with respect to t

3.
   If: $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$
   then: $\dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$
   and: $\ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$

4. If the position vector $\mathbf{r}$ of a particle is given

   then: $\mathbf{v} = \dot{\mathbf{r}} = \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$

   and: $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$

5. The speed of a particle is the magnitude of its velocity vector, that is:

   $$\text{speed} = |\mathbf{v}|$$

Chapter 2 summary - Centres of mass

1. The centre of mass of a lamina is the point at which the weight acts.
   The weight of a uniform lamina is evenly distributed.
   The centre of mass for a lamina or discrete mass distribution must lie on any axis of symmetry.
   The centre of mass of a set of n masses $m_1, m_2, ..., m_n$ at points with coordinates $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ has coordinates $(\bar{x}, \bar{y})$ where:

   $$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

   and: $$\bar{y} = \frac{\sum m_i y_i}{\sum m_i}$$

2. Standard results:

Body	**Centre of mass**
Uniform rod	mid-point of rod
Uniform rectangular lamina	point of intersection of lines joining mid-points of opposite sides
Uniform circular disc	centre of circle
Uniform triangular lamina	point of intersection of medians - in other words $\frac{2}{3}$ distance from any vertex to the mid-point of the opposite side
Circular arc, radius r, angle at centre 2α	$\frac{r \sin \alpha}{\alpha}$ from centre
Sector of circle, radius r, angle at centre 2α	$\frac{2r \sin \alpha}{3\alpha}$ from centre

3. For a lamina which is freely suspended and hangs in equilibrium, the centre of mass will be vertically below the point of suspension.
   For a lamina which is balanced on an inclined plane, the line of action of the weight must fall within the side of the lamina that is in contact with the plane.

Chapter 3 summary - Work, energy and power

1. **Work**
   For a force acting in the direction of the motion:

   Work done = force × distance moved

   For a force acting in a direction other than that of the motion:

   Work done = component of the force in the direction of the motion × distance moved in the same direction

   A force of 1 newton (N) does 1 joule (J) of work when moving a particle a distance of 1 metre.

2. **Energy**
   The kinetic energy (K.E.) of a body of mass m moving with speed $v \mathrm{ms}^{-1}$ is given by:

   $$\text{K.E.} = \frac{1}{2}mv^2$$

   For a mass in kg and velocity in ms^{-1} the K.E. is measured in joules (J). K.E. is never negative.
   The potential energy (P.E.) of a body of mass m at a height h above a chosen level is given by

   $$\text{P.E.} = mgh$$

   Potential energy is also measured in joules. P.E. can be negative.
   The work done on a body is equal to its change of mechanical energy which is K.E. + P.E.

3. **Conservation of energy**
   If the weight of the particle is the only force having a component in the direction of motion then throughout the motion:

   K.E. + P.E. = constant

4. **Power**
   Power is the rate of doing work.
   For a moving particle:

   Power = driving force × speed

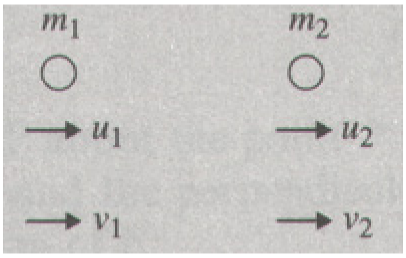
   Power is measured in watts, where 1 watt (W) is 1 joule per second, or kilowatts, where 1 kilowatt (kW) is 1000 watts.

Chapter 4 summary - Collisions

1. **Conservation of linear momentum**
   When two particles collide:
   total momentum before the collision = total momentum after the collision

   $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

   Before collision:
   After collision:

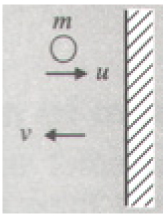
2. **Newton's law of restitution**

   $\frac{\text{speed of separation of particles}}{\text{speed of approach of particles}} = e$

   where e is the coefficient of restitution between the particles.

3. Impact of a particle normally with a fixed surface.

   Before impact:
   After impact:

   (speed of rebound) = e (speed of approach)

   where e is the coefficient of restitution between the particle and the surface.

Chapter 5 summary - Statics of rigid bodies

1. A rigid body is in **equilibrium** if:
   i. the vector sum of the forces acting is zero, that is the sum of the components of the forces in any given direction is zero.
   ii. the algebraic sum of the moments of the forces about any given point is zero.

2. Only in the case of limiting equilibrium, when motion is on the point of taking place, does the frictional force F have its maximum value μR.

M3 summary

M3

Chapter 1 summary - Further kinematics

1. For a particle P moving in a straight line, which at time t seconds has a displacement x metres from a fixed point O of the line, the acceleration $a\mathrm{ms}^{-2}$ and the velocity $v\mathrm{ms}^{-1}$ as functions of time are given by:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

2. When the acceleration is a function of the displacement then:

$$a = v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right)$$

Chapter 2 summary - Elastic springs and strings

1. Elastic springs and strings have a tension or thrust, T, given by:

$$T = \frac{\lambda x}{l}$$

where λ is the modulus of elasticity, x the extension or compression and l the natural length of the string or spring. This is known as Hooke's law.

2. The work done in stretching (or compressing) an elastic string (or spring) with modulus λ from its natural length l to a length $(l + x)$ (or $(l - x)$) is $\frac{\lambda x^2}{2l}$.

3. The elastic energy (E.P.E.) in a string (or spring) of modulus λ extended (or compressed) by a length x beyond its natural length is also $\frac{\lambda x^2}{2l}$.

4. The total change of the mechanical energies (that is kinetic, gravitational potential and elastic potential energies) of a system is equal to the work done by any external forces acting on the system.

Chapter 3 summary - Further dynamics

1. For a particle of mass m moving in a straight line under the influence of a force $F = \mathrm{F}(t)$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{F}(t)$$

2. For a particle of mass m moving in a straight line under the influence of a force $F = \mathrm{G}(x)$

$$m\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \mathrm{G}(x)$$

3. The impulse of a variable force $\mathrm{F}(t)$ acting over the time interval t_1 to t_2 is:

$$\int_{t_1}^{t_2} \mathrm{F}(t)\mathrm{d}t$$

4. The work done by a variable force $\mathrm{G}(x)$ which moves its point of application from x_1 to x_2 is:

$$\int_{x_1}^{x_2} \mathrm{G}(x)\mathrm{d}x$$

5. A particle which moves on a straight line so that its acceleration is always towards a fixed point O in the line and is proportional to its displacement from O is said to move with simple harmonic motion or S.H.M.

   This is written as $\ddot{x} = -\omega^2 x$

   The maximum displacement of the particle from O is the amplitude, a, of the motion.

   The period of the motion is $\frac{2\pi}{\omega}$

   The speed, v, at any point is given by

   $$v^2 = \omega^2(a^2 - x^2)$$

   and $x = a\sin\omega t$ if $x = 0$ when $t = 0$
   $x = a\cos\omega t$ if $x = a$ when $t = 0$
   $x = a\sin(\omega t + \alpha)$ if x is at some other point when $t = 0$

6. A particle attached to the end of an elastic spring which is displaced from its equilibrium position and then allowed to move under the influence of the tension or thrust only will move with S.H.M.
   If the particle is attached to a string, the motion will be simple harmonic only when the string is taut.

Chapter 4 summary - Circular motion

1. When a particle P is moving on a circle centre O and the angle between OP and a fixed radius OA is θ, the angular speed of OP is ω where

   $$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

2. Angular speed is usually measured in radians per second (rad s^{-1}) or revolutions per minute (rev min^{-1}).

3. The linear speed v of the particle is directed along the tangent to the circular path.

4. The linear and angular speeds are connected by the relationship

   $$v = r\omega$$

   where r is the radius of the circular path.

5. When r is measured in metres and ω in radians per second, v is in metres per second.

6. A particle moving in a circular path at a constant speed has an acceleration of magnitude a directed towards the centre of the circle where a is given by

   $a = r\omega^2$ or $a = \frac{v^2}{r}$

7. The resultant force $\mathbf{F}$ acting on a particle of mass m moving on a circular path with constant speed has magnitude $mr\omega^2$ or $\frac{mv^2}{r}$ and is directed towards the centre of the circle.

8. A surface which is inclined at an angle to the horizontal is said to be *banked*.

Motion of a particle in a vertical circle

9. A particle which is moving in a *vertical* circle has variable speed.

10. The acceleration of a particle moving in a circle of radius r with speed v at time t has two components:

    $\frac{v^2}{r}$ towards the centre of the circle

    and $\frac{\mathrm{d}v}{\mathrm{d}t}$ along the tangent.

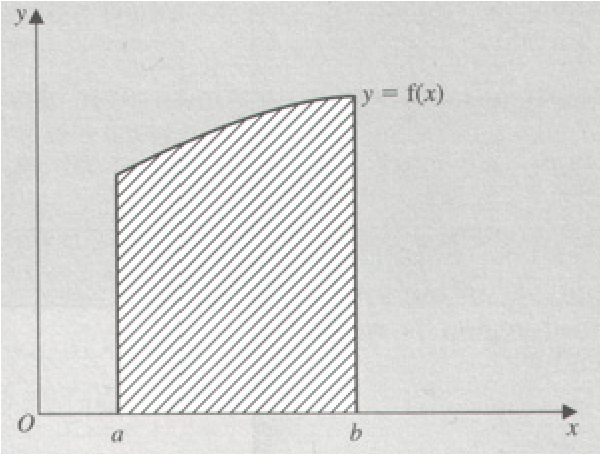
11. The component of the force along the radius does no work during the motion.

12. By the work-energy principle, the sum of the K.E. and P.E. of the particle is constant throughout the motion.

13. A particle which cannot leave its vertical circular path (for example a particle on a rod) will describe complete circles provided its velocity at the highest point of the circle is greater than or equal to zero.

14. A particle which can leave its vertical circular path (for example a particle on a string) will do so when the force towards the centre of the circle becomes zero.

Chapter 5 summary - Statics of rigid bodies

1. 

   The coordinates $(\bar{x}, \bar{y})$ of the centre of mass of the shaded region are given by:

$$\left[\int_a^b \mathrm{f}(x)\mathrm{d}x\right]\bar{x} = \int_a^b x\mathrm{f}(x)\mathrm{d}x$$

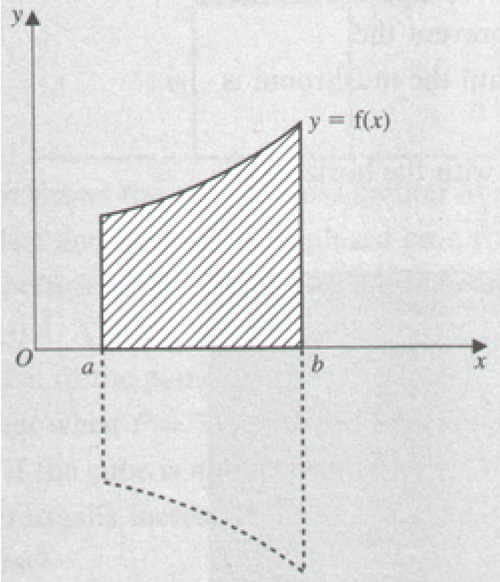
$$\left[\int_a^b \mathrm{f}(x)\mathrm{d}x\right]\bar{y} = \frac{1}{2}\int_a^b [\mathrm{f}(x)]^2\mathrm{d}x$$

2. The centre of mass of a solid body is the point at which the weight acts.
   The weight of a uniform solid body is evenly distributed throughout its volume.
   The centre of mass of a uniform solid body must lie on any **axis** of symmetry.
   The centre of mass of a uniform solid body must lie on any **plane** of symmetry.
   The centre of mass of a system of n masses, $m_1, m_2, \ldots, m_n$ at points with coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$ has coordinates $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

$$\bar{y} = \frac{\sum m_i y_i}{\sum m_i}$$

$$\bar{z} = \frac{\sum m_i z_i}{\sum m_i}$$

3. 

The x-coordinate of the centre of mass, $\bar{x}$, of the solid of revolution generated when the shaded region is rotated through 360° about the x-axis is given by:

$$M\bar{x} = \left[\int_a^b \rho\pi y^2 \mathrm{d}x\right]\bar{x} = \int_a^b \rho\pi y^2 x \mathrm{d}x$$

4. **Standard results for uniform bodies**

Body	**Centre of mass**
Solid hemisphere, radius r	$\frac{3}{8}r$ from centre
Hemispherical shell, radius r	$\frac{1}{2}r$ from centre
Circular arc, radius r, angle at centre 2α	$\dfrac{r\sin\alpha}{\alpha}$ from centre
Sector of circle, radius r, angle at centre 2α	$\dfrac{2r\sin\alpha}{3\alpha}$ from centre
Solid right circular cone, height h	$\dfrac{3h}{4}$ from vertex
Conical shell, height h	$\dfrac{2\lambda}{3}$ from vertex

5. A rigid body is in **equilibrium** if:
   i. the vector sum of the forces acting is zero, that is the sum of the components of the forces in any given direction is zero.
   ii. the algebraic sum of the moments of the forces about any given point is zero.

6. A rigid body hangs in equilibrium with its centre of mass vertically below the point of suspension. For a body in contact with a horizontal or inclined plane to be in equilibrium, the line of action of the weight, through the centre of mass, must lie inside the area of contact.

7. To decide if equilibrium will be broken by sliding or toppling two cases need to be examined:
   i. when the body is on the point of sliding so that $F = \mu R$
   ii. when the body is on the point of toppling, so that the reaction acts at the point about which the body will turn.

S1 summary

Chapter 1 summary - Mathematical modelling in probability and statistics

This chapter does not have a summary.

Chapter 2 summary - Representation of sample data

1. For a **stem and leaf diagram** each row represents a **stem** and is indicated by teh number to the left of the vertical line. The digits to the right of the vertical line are the **leaves** associated with the stem.

2. A **grouped frequency distribution** consists of several **classes** and their associated **class frequencies**.

   For the **class 5-9** for example the

lower class boundary	is	4.5
lower class limit	is	5
upper class limit	is	9
upper class boundary	is	9.5
class width	is	$9.5 - 4.5 = 5$
class mid-point	is	$\frac{1}{2}(4.5 + 9.5) = 7$

3. When drawing a **histogram**, for each histogram bar the area is directly proportional to the frequency that it is representing:

   Area $\propto$ **Frequency**

   and since the histogram consists of a series of bars, then for a histogram:

   Total Area $\propto$ **Total Frequency**

4. The **height** of a histogram bar is found by dividing the class frequency by the class width.

5. Histograms are plotted using class boundaries.

Chapter 3 summary - Methods for summarising sample data (location)

1. The **mode** is that value of a variate which occurs most frequently.

2. The **median** is the middle value of an ordered set of data.

3. The **quartiles** of an ordered set of data are such that 25% of the observations are less than or equal to the first quartile (Q_1), 50% are less than or equal to the second quartile (Q_2) and 75% are less than or equal to the third quartile (Q_3).

4. The **mean** of a set of observations is the sum of all the observations divided by the total number of observations, i.e.

   $$\mu = \bar{x} = \frac{\sum x}{n} \qquad \text{or} \qquad \frac{\sum fx}{\sum f}$$

Chapter 4 summary - Methods for summarising data (dispersion)

1. The range of a data set is given by:

   Range = largest value - smallest value

2. The interquartile range is given by

   $$\mathrm{IQR} = Q_3 - Q_1$$

3. The semi-interquartile range is defined as:

   $$\mathrm{SIQR} = \tfrac{1}{2}(Q_3 - Q_1)$$

4. Variance of a population is defined as:

   $$\sigma^2 = \frac{\sum (x-\mu)^2}{n} \qquad \text{or} \qquad \sigma^2 = \frac{\sum f(x-\mu)^2}{\sum f}$$

5. Unbiased estimator of the population variance is defined as:

   $$s^2 = \frac{\sum (x-\bar{x})^2}{n-1} \qquad \text{or } s^2 = \frac{\sum f(x-\bar{x})^2}{\sum f - 1}$$

6. The standard deviation is the positive square root of the variance.

7. For
   positive skew: $Q_2 - Q_1 < Q_3 - Q_2$
   negative skew: $Q_2 - Q_1 > Q_3 - Q_2$
   symmetry: $Q_2 - Q_1 = Q_3 - Q_2$

Chapter 5 summary - Probability

1. $\mathrm{P}(\text{event } A \text{ or event } B) = \mathrm{P}(A \cup B)$
   $\mathrm{P}(\text{both events } A \text{ and } B) = \mathrm{P}(A \cap B)$
   $\mathrm{P}(\text{not event } A) = \mathrm{P}(A')$

2. **Complementary probability**

   $$\mathrm{P}(A') = 1 - \mathrm{P}(A)$$

3. **Addition rule**

   $$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

4. **Conditional probability**

   $$\mathrm{P}(A \text{ given } B) = \mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

5. **Multiplication rule**

   $$\mathrm{P}(A \cap B) = \mathrm{P}(A|B) \times \mathrm{P}(B)$$

6. A and B are **independent** events if

   $$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$$

7. A and B are **mutually exclusive** events if

   $$\mathrm{P}(A \cap B) = 0$$

Chapter 6 summary - Correlation

1. Product-moment correlation coefficient:

   $$r = \frac{S_{xy}}{\sqrt{S_{xx}S_y y}}$$

   where

   $$S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \frac{\sum x_i \sum y}{n}$$

   $$S_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n}$$

   $$S_{yy} = \sum (y_i - \bar{y})^2 = \sum y_i^2 - \frac{(\sum y_i)^2}{n}$$

2. r is a measure of linear association

   $r = 1 \Rightarrow$ perfect positive linear correlation
   $r = -1 \Rightarrow$ perfect negative linear correlation
   $r = 0 \Rightarrow$ no linear correlation

Chapter 7 summary - Regression

1. **Explanatory or independent variable:**
   a variable that is set independently of the other variable

2. **Response or dependent variable:**
   the variable whose values are decided by the values of the explanatory or independent variable.

3. **Linear regression model:**

   $$y_1 = \alpha + \beta x_i + \varepsilon_i$$

4. **The regression line of y on x is:**

   $$y = a + bx,$$

   where $b = \dfrac{S_{xy}}{S_{xx}}$ and $a = \bar{y} - b\bar{x}$

Chapter 8 summary - Discrete random variables

1. For a **discrete random variable X**

   $$\sum_{\forall x} \mathrm{P}(X = x) = 1$$

   $$\mu = \mathrm{E}(X) = \sum_{\forall x} x\mathrm{P}(X = x)$$

   $$\sigma^2 = \mathrm{E}(X^2) - \mu^2 = \sum_{\forall x} x^2\mathrm{P}(X = x) - \mu^2$$

2. **Properties of expected values and variance**

   $$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

   $$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

3. **Cumulative distribution function $\mathrm{F}(x)$**

   $$0 \le \mathrm{F}(x) \le 1$$

4. **For the *discrete* random variable X:**

   $$\mathrm{F}(x_0) = \mathrm{P}(X \le x_0) = \sum_{x \le x_0} \mathrm{P}(X = x)$$

Chapter 9 summary - The normal distribution

1. For a continuous random variable X, having a normal distribution,

   Mean $= \mu$
   Variance $= \sigma^2$

2. Given that $X \sim \mathrm{N}(\mu, \sigma^2)$,

   then $\quad Z = \dfrac{X - \mu}{\sigma} \sim \mathrm{N}(0, 1^2)$

Chapter 1 summary - The binomial and Poisson distributions

1. **Binomial distribution**

   $X \sim B(n, p)$

   $$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r} \qquad r = 0, 1, \ldots n$$

   $\mu = E(X) = np$

   $\sigma^2 = \text{Var}(X) = np(1-p) = npq$

2. **Poisson distribution**

   $X \sim \text{Po}(\lambda)$

   $$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}$$

   $\mu = E(X) = \lambda$

   $\sigma^2 = \text{Var}(X) = \lambda$

3. **Poisson approximation to binomial**

   $X \sim B(n, p)$

   if n is large
   and p is small
   then $X \approx\sim \text{Po}(np)$

Chapter 2 summary - Continuous random variables

1. **Continuous random variable X**

$$\int_{-\infty}^{\infty} \mathrm{f}(x)\mathrm{d}x = 1$$

$$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x\mathrm{f}(x)\mathrm{d}x$$

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2 = \int_{-\infty}^{\infty} x^2\mathrm{f}(x)\mathrm{d}x - \mu^2$$

2. **Cumulative distribution function $\mathrm{F}(x)$**

$$0 \leq \mathrm{F}(X) \leq 1$$

$$\mathrm{F}(x_0) = \mathrm{P}(X \leq x_0) = \int_{-\infty}^{x_0} \mathrm{f}(x)\mathrm{d}x$$

Median m satisfies $\mathrm{F}(m) = 0.5$
Quartile Q_1 satisfies $\mathrm{F}(Q_1) = 0.25$
Quartile Q_3 satisfies $\mathrm{F}(Q_3) = 0.75$

Chapter 3 summary - Continuous distributions

1. A random variable X, having a continuous uniform distribution over the interval (α, β) has p.d.f.

$$\mathrm{f}(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \alpha < x < \beta \\ 0, & \text{otherwise.} \end{cases}$$

2. For a random variable X, having a uniform distribution

$$\mathrm{E}(X) = \frac{\alpha+\beta}{2}$$

$$\mathrm{Var}(X) = \tfrac{1}{12}(\beta-\alpha)^2$$

3. A random variable $X \sim \mathrm{B}(n,p)$ can be approximated by $Y \sim \mathrm{N}(\mu, \sigma^2)$ when $\mu = np$ and $\sigma^2 = np(1-p) = npq$ provided that n is large, $np > 5$ and $n(1-p) = nq > 5$.

4. A random variable $X \sim \mathrm{Po}(\lambda)$ can be approximated by

$Y \sim \mathrm{N}(\lambda, \lambda)$ for $\lambda > 10$

Chapter 4 summary - Hypothesis tests

1. A **population** is a collection of individual items.
2. A **sample** is a selection of individual members or items from a population.
3. A **finite population** is one in which each individual member can be given a number.
4. An **infinite population** is one in which it is impossible to number each member.
5. A **countably infinite population** is one which is infinite in size, but each member can be given an individual number.
6. A **sampling unit** is an individual member of a population.
7. A **sampling frame** is a list of sampling units used in practice to represent a population. In some instances the two will be identical, in others the sampling frame will represent the population as accurately as possible.
8. In practice a **sample** is a collection of sampling units drawn from a sampling frame.
9. A **hypothesis test** is a mathematical procedure to examine a value of a population parameter proposed by the null hypothesis H_0 compared with an alternative hypothesis H_1.
10. The **critical region** is the range of values of a test statistic T that would lead you to reject H_0.

Chapter 1 summary - Algorithms

1. An **algorithm** is a set of precise instructions which if followed will solve a problem.

2. **The bubble-sort algorithm**
   To sort a list compare adjacent members of the list, moving from left to right, and switch them if they are in the wrong order. Continue this process until a pass produces no change in the list.

3. **The quick-sort algorithm**

   Step 1 Select a specific number (the pivot) from the list, say the middle one.

   Step 2 Write all the numbers smaller than the pivot to the left of the pivot, reading the original list from left to right, and so create a sublist L_1. Write all the numbers larger than the pivot to the right of the pivot, reading the original list from left to right, and so create a sublist L_2.

   Step 3 Apply step 1 and 2 to each sublist until all the sublists contain only one number.

4. **First-fit algorithm**
   Taking the boxes in the order listed place the next box to be packed in the *first* available bin that can take that box.

5. **First-fit decreasing algorithm**

   Step 1 Reorder the boxes in *decreasing* order of size.

   Step 2 Apply the first-fit algorithm to the reordered list.

6. **Binary-search algorithm**
   May be applied to search a list of names in alphabetical order or a list of numbers in ascending order.

   Step 1 Compare the required item with the middle item in the list. If this is the required item then the search is complete.
   i. If the required item is before the middle item then consider the top half of the list.
   ii. If the required item is after the middle item then consider the bottom half of the list.

   Step 2 Apply step 1 to the top half of the list in case (i) or to the bottom half of the list in case (ii).

   Stop Either when the required item is located or when it has been shown that the required item is not in the list.

Chapter 2 summary - Graphs and networks

1. A **graph G** consists of a finite number of points (usually called vertices or nodes) connected by lines (usually called edges or arcs).
2. A **path** is a finite sequence of edges such that the end vertex of one edge in the sequence is the start vertex of the next, and in which no vertex appears more than once.
3. A **cycle** (or circuit) is a closed path, i.e. the end vertex of the last edge is the start vertex of the first edge.
4. A **Hamiltonian cycle** is a cycle that passes through every vertex of the graph once and only once, and returns to its start vertex.
5. A **Eulerian cycle** is a cycle that includes every edge of a graph exactly once.
6. The **vertex set** is the set of all vertices of a graph.
7. The **edge set** is the set of all edges of a graph.
8. A **subgraph** of a graph is a subset of the vertices together with a subset of edges.
9. Two vertices are **connected** if there is a path between them.
10. A graph is **connected** if all pairs of its vertices are connected.
11. A **simple graph** is one in which there is no edge with the same vertex at each end, i.e. no loops, and not more than one edge connecting any pair of vertices. A **complete graph** is one in which every vertex is connected by an edge to each of the other vertices.
12. The **degree** (or valency or order) of a vertex is the number of edges connected to it.
13. If the edges of a graph have a direction associated with them they are known as **directed edges**, and the graph is known as a **digraph**.
14. A **tree** is a connected graph with no cycles.
15. A **spanning tree** of a graph G is a subgraph that includes all the vertices of G and is also a tree.
16. A **bipartite graph** consists of two sets of vertices, X and Y. The edges only join vertices in X to vertices in Y, not vertices within a set.
17. If there are r vertices in X and s vertices in Y and every vertex in X is joined to every vertex in Y then the graph is called $\mathrm{K}_{r,s}$.

Chapter 3 summary - Algorithms on graphs

1. A **minimum spanning tree** of a connected and undirected graph is a spanning tree such that the total length of its edges is as small as possible. (This is sometimes called a minimum connector.)

2. **Kruskal's algorithm** builds a minimum spanning tree by adding one **edge** at a time to build a subgraph. At each stage the edge of smallest weight is chosen provided that it does not create a cycle with edges already chosen.

3. **Prim's algorithm** builds a minimum spanning tree by adding one **vertex** at a time to a connected subgraph. The new vertex to be added is the one which is nearest to any vertex already in the subgraph.

4. **Dijkstra's algorithm** obtains the shortest route from the initial vertex to any other vertex in a network. At each stage a fresh vertex is assigned a **final label** which gives the shortest distance from the initial vertex to that vertex.

Chapter 4 summary - The route inspection problem

1. A **traversable graph** is one that can be drawn without removing your pen from the paper and without going over the same edge twice.

2. If a graph is not Eulerian but there is a route starting and finishing at different vertices and traversing every edge once and only once then the graph is **semi-Eulerian**.

3. If in a graph there is a route that starts and finishes at the same vertex and traverses every edge once and only once then the graph is **Eulerian**.

4. **The handshaking theorum**
   The **sum** of the valencies, taken over all the vertices of a graph G, is equal to **twice** the number of edges.

5. **The route inspection problem**
   In a given undirected network a route of **minimum weight** has to be found that traverses every edge at least once, returning to the starting vertex.

Chapter 5 summary - Critical path analysis

In an activity network:

1. The **nodes** represent events.
2. The **arcs** represent activities.
3. The **weights** represent the duration of the corresponding activity.
4. The **source node** represents the beginning of the project.
5. The **sink node** represents the end of the project.
6. The **earliest time** e_i for node i is the earliest time we can arrive at event i.
7. The **latest time** l_i for node i is the latest time we can leave event i without extending the length of the critical path.
8. The **critical path** is the longest path through the network from the source node to the sink node.
9. Events i for which $e_i = l_i$ are **critical events**.
10. Activities (i, j) for which $l_j - e_i - [\text{duration}(i, j)] = 0$ are **critical activities**.
11. For activity (i, j) of duration a_{ij}:

    the earliest start time is e_i
    the earliest finish time is $e_i + a_{ij}$
    the latest finish time is l_j
    the latest start time is $l_j - a_{ij}$
    the total float on activity (i, j) is $l_j - e_i - a_{ij}$

Chapter 6 summary - Linear programming

1. Any pair of values of x and y that satisfy all the constraints in a linear programming problem is called a **feasible solution**.
2. The region that contains all feasible solutions is called the **feasible region**.
3. The **optimal solution** of a linear programming problem, if it exists, will occur at one or more of the extreme points (vertices) of the feasible region.
4. The **simplex method** is an algebraic method for solving linear programming problems.
    i. The column that contains the entering variable is called the **pivotal column**.
    ii. The row with the smallest θ-value is called the **pivotal row**.
    iii. The entry at the intersection of the pivotal row and the pivotal column is called the pivot.
5. **Optimality condition:**
   If the objective row of a tableau has zero entries in the columns labelled by basic variables and no negative entries in the columns labelled by non-basic variables then the solution represented by the tableau is optimal.

Chapter 7 summary - Matchings

1. A **bipartite graph** consists of two sets of vertices X and Y. The edges only join vertices in X to vertices in Y, not vertices within a set.

2. A **matching** in a bipartite graph is a subset M of the edges of the graph G such that no two edges in M have a common vertex.

3. A **maximal matching** M is a matching in which the number of edges is as large as possible.

4. In a bipartite graph, with n vertices in each set a **complete matching** M is a matching in which the number of edges is also n.

5. An **alternating path** for a matching M in G is a path in G with the following properties:
   i. it joins an unmatched vertex in X to an unmatched vertex in Y
   ii. it is such that edges in the path are alternately **in** and **not in** the matching M.

6. **The matching improvement algorithm**

   Step 1 Start with any non-trivial matching M in G.

   Step 2 Search for an alternating path for M in G.

   Step 3 If an alternating path is found, construct a better matching M' by changing the status of the edges in the alternating path and return to step 2 with M' replacing M.

   Step 4 Stop when no alternating path can be found. The matching obtained is maximal.

Chapter 8 summary - Flows in networks

1. A **network** is a weighted digraph in which the weight on each arc represents the capacity of that arc.

2. A vertex S is called a **source** if all arcs containing S are directed away from S.

3. A vertex T is called a **sink** if all arcs containing T are directed towards T.

4. A **flow** in a network N with a single source S and a single sink T is an assignment to each arc e of N of a non-negative number called the flow along the arc e.
   A flow must satisfy two conditions:
   - The **feasibility condition**, which states that the flow along each arc cannot exceed the capacity of that arc.
   - The **conservation condition**, which states that for every intermediate vertex V (not S or T), the sum of the flows into V is equal to the sum of the flows out of V.
5. If the flow along an arc is equal to the capacity of that arc the arc is said to be **saturated**. If an arc is not saturated it is said to be **unsaturated**.

6. A **flow-augmenting path** in a capacitated network with a single source S and a single sink T is a path from S to T consisting of:
   - **forward arcs** - unsaturated arcs directed along the path.
   - **backward arcs** - arcs directed against the direction of the path and carrying a non-zero flow.

7. In any capacitated network with a single source S and a single sink T, **the value of a maximal flow is equal to the capacity of a minimum cut**.

8. The maximum flow algorithm is as follows:

   Step 1 Obtain an initial flow by inspection.

   Step 2 Find flow-augmenting paths using the labelling procedure until no further flow-augmenting paths can be found.

   Step 3 Check that the flow obtained is maximal by finding a cut whose capacity is equal to the value of the flow.

9. A network with multiple sources $\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_m$ and multiple sinks $\mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_n$ can be reduced to a network with only one source and one sink by introducing a **supersource** S and a **supersink** T. Arcs $\mathrm{SS}_1, \mathrm{SS}_2, \ldots, \mathrm{SS}_m$ are added from the supersource S to the sources $\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_m$. Arcs $\mathrm{T}_1\mathrm{T}, \mathrm{T}_2\mathrm{T}, \ldots, \mathrm{T}_n\mathrm{T}$ are added from sinks $\mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_n$ to the supersink T. The capacities of the new arcs can be considered to be infinite.